83. *Boundedness of Closed Linear Operator T satisfying* $R(T)\subset D(T)$

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1. Let T be a densely defined closed linear operator in a Banach space E satisfying that $R(T)\subset D(T)$. We prove that if T satisfies one of the following conditions:

(1) $\|T^2x\|\|x\|\geqq\|Tx\|^2$ for every $x\in D(T)$, or

(2) T has the non-empty resolvent set,

then it follows that T is bounded.

Let $T: E\to H$ be a densely defined closed linear operator and $T^*: H\to E'$ be the adjoint operator, where H is a Hilbert space. It is shown that if $R(T)\subset D(T^*)$, then T is bounded.

2. Let E be a Banach space and $T: E\to E$ be a densely defined closed linear operator with the domain $D(T)$ and the range $R(T)$. The following problem was posed by Ôta [3].

Problem. *Suppose that $R(T)\subset D(T)$, then is T bounded?*

In general, the answer is negative as shown by Ôta [3]. Ôta [3] proved that if T is dissipative, then the answer is positive. In this note we investigate other conditions which imply the positive answers for this problem.

After Furuta [1], we say the linear operator $T: E\to E$ *paranormal* if $R(T)\subset D(T)$ and if it holds that $\|T^2x\|\|x\|\geqq\|Tx\|^2$ for every $x\in D(T)$.

Theorem 1. *Let T be a densely defined closed paranormal operator in a Banach space E. Then T is bounded.*

Proof. Since T is closed, $(D(T), |\ |_T)$ is a Banach space, where $|x|_T=\|x\|+\|Tx\|$, $x\in D(T)$. By $R(T)\subset D(T)$, the operator $T^2: (D(T), |\ |_T)\to E$ is well defined. By the closedness of T, it follows that T^2 is also closed on $(D(T), |\ |_T)$, hence bounded. Thus there exists $C>0$ such that $\|T^2x\|\leqq C(\|x\|+\|Tx\|)$ for every $x\in D(T)$. By the paranormality, we have for every $x\in D(T)$ with $\|x\|=1$, $\|Tx\|^2\leqq\|T^2x\|\leqq C(1+\|Tx\|)$. That is, $\|Tx\|^2-C\|Tx\|-C\leqq 0$. This implies that

$$\|Tx\|\leqq\frac{C+\sqrt{C^2+4C}}{2}<+\infty,$$

which implies the assertion.

Let T be a linear operator in a Banach space E. The *resolvent set* $\rho(T)$ of T is the set of all complex numbers λ such that the range $R(\lambda I-T)$ is dense in E and that $\lambda I-T$ has the continuous inverse $(\lambda I-T)^{-1}$ on $D((\lambda I-T)^{-1})=R(\lambda I-T)$, see Yosida [4], Ch. VIII. It is well known that if T is bounded, then $\rho(T)\neq\emptyset$. The converse is valid for a densely defined closed linear operator satisfying $R(T)\subset D(T)$.

Theorem 2. *Let T be a densely defined closed linear operator in a Banach space E satisfying that $R(T)\subset D(T)$. Suppose that $\rho(T)\neq\emptyset$, then T is bounded.*

Proof. Take $\lambda\in\rho(T)$. Since $\lambda I-T$ has the continuous inverse, there exists $C>0$ such that $\|(\lambda I-T)x\|\geqq C\|x\|$ for every $x\in D(\lambda I-T)=D(T)$. By the closedness of T, it follows that the range $R(\lambda I-T)$ is closed. In fact, let $(\lambda I-T)x_n\to z$, $x_n\in D(T)$ and $z\in E$. Then by this inequality, $\{x_n\}$ is a Cauchy sequence in E, hence $x_n\to x$ for some $x\in E$. Thus we have $x_n\to x$ and $Tx_n\to\lambda x-z$. By the closedness of T, we have $x\in D(T)$ and $Tx=\lambda x-z$, which shows the assertion. Since $\lambda\in\rho(T)$, $R(\lambda I-T)$ is dense in E, so $R(\lambda I-T)=E$. Consequently it follows that $E=R(\lambda I-T)\subset D(T)$, that is, $D(T)=E$. Thus T is bounded.

Theorem 3. *Let T be a densely defined closed linear operator in a Banach space E satisfying that $R(T)\subset D(T)$. Suppose that there exists $N>0$ such that for every $n>N$, there exists $K_n>0$ such that $\|(nI-T)x\|\geqq K_n\|x\|$, $x\in D(T)$. Then T is bounded.*

Proof. We shall show that $D(T)=E$. By $R(T)\subset D(T)$, the operator $T:(D(T),|\ |_T)\to(D(T),|\ |_T)$ is well defined and bounded as easily seen, $|x|_T=\|x\|+\|Tx\|$ for $x\in D(T)$. There exists $C>0$ such that $|Tx|_T\leqq C|x|_T$ for every $x\in D(T)$. Let n be $n>N$ and $n>C$. Then it follows that $nI-T$ has the bounded inverse $(nI-T)^{-1}$ which is everywhere defined on $(D(T),|\ |_T)$. Thus we have $D((nI-T)^{-1})=R(nI-T)=D(T)$. Since $\|(nI-T)x\|\geqq K_n\|x\|$, $x\in D(T)$, by the manner same to Theorem 2, we can see that the range $R(nI-T)$ is closed. Consequently it follows that $E=\overline{D(T)}=R(nI-T)=D(T)$, which proves the assertion.

3. Let H be a Hilbert space. Ôta [3] proved that if T is a densely defined closed linear operator in H satisfying that $R(T)\subset D(T^*)$, then T is bounded. We shall prove an analogous result for an operator $T:E\to H$, where E is a Banach space and H is a Hilbert space.

Let $T:E\to H$ be a densely defined linear operator. The *adjoint* T^* of T is defined by $D(T^*)=\{y\in H;\ D(T)\ni x\to(Tx,y)\text{ is continuous}\}$ and $(T^*y)(x)=(Tx,y)$ for $x\in D(T)$ and $y\in D(T^*)$. The adjoint T^* is an operator in H into E'.

Theorem 4. *Let E be a Banach space, H be a Hilbert space and $T:E\to H$ be a densely defined closed linear operator. If $R(T)\subset D(T^*)$, then T is bounded.*

Proof. Since T is closed, $(D(T),|\ |_T)$ is a Banach space, where $|x|_T=\|x\|_E+\|Tx\|_H$. By $R(T)\subset D(T^*)$, the operator $T^*T:(D(T),|\ |_T)\to E'$ is well defined. Remarking that T^* is closed since $D(T)$ is dense, we can see that T^*T is also closed on $(D(T),|\ |_T)$, hence bounded. There exists $C>0$ such that $\|T^*Tx\|_{E'}=C(\|x\|_E+\|Tx\|_H)$ for every $x\in D(T)$. For every $x\in D(T)$ with $\|x\|_E=1$, it follows that $\|T^*Tx\|_{E'}\geqq|(T^*Tx)(x)|=|(Tx,Tx)|=\|Tx\|_H^2$ and hence $\|Tx\|_H^2\leqq C(1+\|Tx\|_H)$. Consequently we have for every

$x \in D(T)$ with $\|x\|_E = 1$,

$$\|Tx\|_H \leqq \frac{C+\sqrt{C^2+4C}}{2} < +\infty,$$

which shows the boundedness of T.

Theorem 5. *Let E be a Banach space, H be a Hilbert space and $T: H \to E$ be a densely defined closable linear operator. If $R(T') \subset D(T)$, then T is bounded, where $T': E' \to H$ is the conjugate operator of T given by $D(T') = \{\xi \in E';\ D(T) \ni x \to \langle Tx, \xi\rangle$ is continuous$\}$ and $(T'\xi)(x) = \langle Tx, \xi\rangle$ for $x \in D(T)$ and $\xi \in D(T')$.*

Proof. Remark that $R(\bar{T}') = R(T') \subset D(T) \subset D(\bar{T})$, where $\bar{T}$ is the closure of T. By the manner same to Theorem 4, it follows that $\bar{T}'$ is continuous on $D(\bar{T}')$. Remark that the density of $D(\bar{T}')$ in E' is not assumed in advance. But by Goldberg [2], Corollary II. 4.8, it follows that $D(\bar{T}) = H$ and $\bar{T}$ is bounded since $\bar{T}$ is closed.

Corollary. *Let $T: H \to H$ be a densely defined closable linear operator in a Hilbert space H. If T satisfies that $R(T^*) \subset D(T)$, then T is bounded.*

References

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