81. *Periodicity and Almost Periodicity of Solutions to Free Boundary Problems in Hele-Shaw Flows*

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This paper is concerned with the asymptotic behavior of solutions to the following problem : given f, g_0, g_1 and u_0, find u such that

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}-\Delta v=f, & v\in\beta(u) \quad \text{in } (0,\infty)\times\Omega \\ v=g_0 & \text{on } (0,\infty)\times\Gamma_0 \\ \dfrac{\partial v}{\partial n}+p\cdot v=g_1 & \text{on } (0,\infty)\times(\Gamma\backslash\Gamma_0) \\ u(0)=u_0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ with smooth boundary Γ, Γ_0 is a compact subset of Γ with positive surface measure, p is a positive bounded measurable function on Γ and β is a maximal monotone graph in $\boldsymbol{R}\times\boldsymbol{R}$. In [6] and [7], the global behavior of solutions to (1) is studied in case when β is Lipschitz continuous. This case corresponds to a Stefan problem in a weak sense. But, for instance, in the weak formulations of free boundary problems arising from Hele-Shaw flows and electro-chemical machining processes, β is in general multi-valued (cf. [3, 8, 12, 13, 14, 15]). In [10] and [11], the stability of solutions to general evolution equations generated by time-dependent subdifferentials is studied. But their results do not seem to be directly applicable to our problem. In this paper, we extend a part of the results in [7] to a class of β including the case of Hele-Shaw flows and electro-chemical machining processes.

Let us use the notations : $H=L^2(\Omega)$ with inner product $(\cdot,\cdot)_H$. And put $V=\{z\in H^1(\Omega)\,;\, z=0 \text{ a.e. on } \Gamma_0\}$. Then V becomes a Hilbert space with inner product

$$(z,y)_V=\int_\Omega \nabla z\cdot\nabla y\,\mathrm{d}x+\int_\Gamma p(x)z(x)y(x)\,\mathrm{d}\Gamma \qquad \text{for } z,y\in V.$$

We denote by V^* the dual space of V and regard V^* as a Hilbert space with inner product $(z,y)_*=\langle z, F^{-1}y\rangle_{V^*,V}$ and norm $|z|_*=\langle z,z\rangle_*^{1/2}$ where $\langle\cdot,\cdot\rangle_{V^*,V}$ is the duality between V^* and V and F is the duality mapping from V onto V^*.

Definition 1. For given constants $a>0$ and $b\geqq 0$, let $B(a,b)$ be the set of all maximal monotone graph β in $\boldsymbol{R}\times\boldsymbol{R}$ such that $\beta=\partial\hat{\beta}$ for some $\hat{\beta}:\boldsymbol{R}\to\boldsymbol{R}\cup\{\infty\}$ proper l.s.c. (lower-semicontinuous) convex function with $\hat{\beta}(0)=0$ and $\hat{\beta}(r)\geqq a|r|^2-b$ for all $r\in\boldsymbol{R}$.

In the problem associated with Hele-Shaw flows, β is the inverse of the Heaviside graph, so that $\beta \in B(1,1)$.

Given $\beta \in B(a,b)$ and $g \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}; H)$, we define a function φ^t on V^* for each $t \in \boldsymbol{R}$ by

$$(2) \qquad \varphi^t(z)=\begin{cases} \int_\Omega \hat{\beta}(z(x))\,\mathrm{d}x-(g(t),z)_H & \text{if } z \in H \\ \infty & \text{if } z \in V^* \backslash H. \end{cases}$$

Lemma 1 (cf. [4, 5]). *For each $t \in \boldsymbol{R}$, φ^t is a proper l.s.c. convex function on V^* with $D(\varphi^t)=\{z \in H; \hat{\beta}(z) \in L^1(\Omega)\}$ and for $u, u^* \in V^*$, $u^* \in \partial\varphi^t(u)$ if and only if the following conditions hold:*

(i) $u \in D(\varphi^t)$.

(ii) *There exists $v \in H$ such that $v-g(t) \in V$, $u^*=F(v-g(t))$ and $v(x) \in \beta(u(x))$ for* a.e. $x \in \Omega$.

Now consider the nonlinear evolution equation in V^*:

$$(3) \qquad u'(t)+\partial\varphi^t(u(t)) \ni f(t), \qquad \text{for a.e. } t \in \boldsymbol{R}_+.$$

On account of Lemma 1, this expression is nothing but the variational formulation of problem (1), provided that we take as $g(t,\cdot)$ the function determined by g_0 and g_1 in a suitable way (see [4, 5]).

Definition 2. Let β and g be as above and let $f \in L^2_{\mathrm{loc}}(\boldsymbol{R}; V^*)$. Then $u: J=[t_0, t_1] \to V^*$ is called a solution to $E(\beta, g, f)$ on J, if it satisfies the following conditions:

(a) $u \in C(J; V^*) \cap W^{1,2}_{\mathrm{loc}}((t_0, t_1]; V^*)$.

(b) $t \mapsto \varphi^t(u(t))$ is in $L^1(J)$, where φ^t, $t \in \boldsymbol{R}$ is given by (2).

(c) (3) holds.

Also for general interval J in $\boldsymbol{R}$, $u: J \to V^*$ is called a solution to $E(\beta, g, f)$, if it is a solution to $E(\beta, g, f)$ on every compact subinterval of J in the above sense.

By virtue of the general existence-uniqueness result (cf. [9]), we have

Lemma 2. *For given $\beta \in B(a,b)$, $g \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}; H)$, $f \in L^2_{\mathrm{loc}}(\boldsymbol{R}; V^*)$ and u_0 in the closure of $\{z \in H; \hat{\beta}(z) \in L^1(\Omega)\}$ in V^*, there exists a unique solution u to $E(\beta, g, f)$ on $\boldsymbol{R}_+$ with $u(0)=u_0$ such that $u \in L^\infty_{\mathrm{loc}}((0,\infty); H)$ and $t \mapsto \varphi^t(u(t))$ is in $W^{1,1}_{\mathrm{loc}}((0,\infty))$.*

We are interested in the periodic and almost periodic behavior of solutions to $E(\beta, g, f)$ on $\boldsymbol{R}$. We shall state only the results about the almost periodicity of solutions. In case when f and g are periodic with the same period T, the corresponding results are obtained as corollaries of them, because in this case every V^*-bounded solution on $\boldsymbol{R}$ must be T-periodic (cf. [7]). For the periodic case, also see [8].

Theorem. *Let $\beta \in B(a,b)$, $g \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}; H)$ be an H-almost periodic function and $f \in L^2_{\mathrm{loc}}(\boldsymbol{R}; V^*)$ be a V^*-almost periodic function in the sense of Stepanov* (cf. [1]). *Suppose that $\sup_{t\in\boldsymbol{R}} |g'|_{L^1(t,t+1;H)} < \infty$ and that if $\{t_n\}$ is a sequence in $\boldsymbol{R}$ and if $g(t+t_n) \to \hat{g}(t)$ in H uniformly in $t \in \boldsymbol{R}$ as $n \to \infty$, then $\hat{g} \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}; H)$ and $\sup_{t\in\boldsymbol{R}} |\hat{g}'|_{L^1(t,t+1;H)} < \infty$. Then we have:*

(i) *$AP \equiv \{u; u$ is a V^*-almost periodic solution to $E(\beta, g, f)$ on $\boldsymbol{R}\} \neq \phi$.*

(ii) *For each solution u to $E(\beta, g, f)$ on $\boldsymbol{R}_+$, there exists $\omega \in AP$ such that $u(t)-\omega(t)\to 0$ in V^* and weakly in H as $t\to\infty$.*

(iii) *Let $\omega_1, \omega_2 \in AP$ and let $\omega_i + F(\eta_i - g) = f$, $\eta_i \in \beta(\omega_i)$* a.e. *on $\boldsymbol{R}$, $i=1, 2$. Then $\eta_1(t)=\eta_2(t)$ for* a.e. *$t\in \boldsymbol{R}$ and there is an element α in H such that $\omega_1(t) = \omega_2(t)+\alpha$ for all $t\in\boldsymbol{R}$.*

(iv) *A solution u to $E(\beta, g, f)$ on $\boldsymbol{R}$ is V^*-almost periodic if and only if u is V^*-bounded on $\boldsymbol{R}$* i.e. $\sup_{t\in \boldsymbol{R}}|u(t)|_*<\infty$.

The proof is similar to that in [7]. The different point is the following Lemma.

Lemma 3. *Let β, g and f be as in the Theorem. Suppose that u_i $(i=1, 2)$ are solutions to $E(\beta, g, f)$ on an interval J. Assume that*

$$t\mapsto |u_1(t)-u_2(t)|_* \quad \text{is constant on } J. \tag{4}$$

Then we have

$$u_1'(t)=u_2'(t) \quad \text{for a.e. } t\in J. \tag{5}$$

Proof. We use the technique in (2). Let φ^t, $t\in\boldsymbol{R}$ be given by (2) and let $u_i'+F(v_i-g)=f$ and $v_i\in\beta(u_i)$ a.e. on J $(i=1, 2)$. Then we have

$$\begin{aligned}
&\varphi^{t+h}(u_i(t+h))-\varphi^t(u_i(t))\\
&\quad=\int_\Omega \hat{\beta}(u_i(t+h))\,\mathrm{dx}-\int_\Omega \hat{\beta}(u_i(t))\,\mathrm{dx}-(g(t+h), u_i(t+h))_H+(g(t), u_i(t))_H\\
&\quad\geqq(v_i(t), u_i(t+h)-u_i(t))_H-(g(t+h), u_i(t+h))_H+(g(t), u_i(t))_H\\
&\quad=\langle v_i(t)-g(t), u_i(t+h)-u_i(t)\rangle_{V^*,V}-(g(t+h)-g(t), u_i(t+h))_H\\
&\quad=(f(t)-u_i'(t), u_i(t+h)-u_i(t))_*-(g(t+h)-g(t), u_i(t+h))_H, \qquad i=1, 2.
\end{aligned}$$

Since u_i, $i=1, 2$ are weakly continuous in H, we obtain by dividing both sides of the above inequality by h and letting $h\to 0$,

$$\frac{d}{dt}\{\varphi^t(u_i(t))\}=(f(t)-u_i'(t), u_i'(t))_*-(g'(t), u_i(t))_H \quad \text{for a.e. } t\in J,\ i=1, 2. \tag{6}$$

On the other hand, from (4) and [11; Lemma 4.4], we have

$$f(t)-u_2'(t)\in\partial\varphi^t(u_1(t)) \quad\text{and}\quad f(t)-u_1'(t)\in\partial\varphi^t(u_2(t)) \quad \text{for a.e. } t\in J.$$

From this fact we have similarly

$$\frac{d}{dt}\{\varphi^t(u_1(t))\}=(f(t)-u_2'(t), u_1'(t))_*-(g'(t), u_1(t))_H \quad \text{for a.e. } t\in J \tag{7}$$

and

$$\frac{d}{dt}\{\varphi^t(u_2(t))\}=(f(t)-u_1'(t), u_2'(t))_*-(g'(t), u_2(t))_H \quad \text{for a.e. } t\in J. \tag{8}$$

Combining (6), (7) and (8) we obtain (5). q.e.d.

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