93. *A Remark on the Essential Self-adjointness of Dirac Operators*

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(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1986)

In this paper we shall consider the essential self-adjointness of Dirac operators

$$H=\sum_{j=1}^{3}\alpha_j D_j+\beta+Q(x),\quad x\in \boldsymbol{R}^3,\quad D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$$

defined on $[C_0^\infty(\boldsymbol{R}^3)]^4$, where α_j and $\alpha_4=\beta$ are 4×4 Hermitian symmetric matrices satisfying

$$\alpha_j^2=I,\quad \alpha_j\alpha_k+\alpha_k\alpha_j=0\quad (j\neq k)$$

(I is the unit matrix). We define α_r by

$$\alpha_r=\sum_{j=1}^{3}(x_j/r)\alpha_j\qquad (r=|x|)$$

which is Hermitian symmetric for each $x\neq 0$ and satisfies

(1) $$\alpha_r^2=I$$

in view of the above anti-symmetric relations. The potential $Q(x)$ is a 4×4 Hermitian symmetric matrix valued function of the following form

$$Q(x)=\frac{ib_1}{r}\alpha_r\beta+\frac{b_2}{r}\beta+V(x),$$

where b_1, b_2 are real constants. M. Arai [1], Theorem 3.1, shows that H is essentially self-adjoint and that the domain of the closure $\overline{H}$ coincides with the Sobolev space $[H^1(\boldsymbol{R}^3)]^4$, if

(2) $$r\left|V(x)+\frac{i}{2r}\alpha_r\right|\leqq m$$

for a positive constant m such that

(3) $$m<m_0\equiv \min_{k\in \boldsymbol{Z}\setminus\{0\}}\sqrt{(k+b_1)^2+b_2^2}$$

(see our Remark 8), where $|A|$ for a matrix A denotes the square root of the largest eigenvalue of A^*A. Moreover, Arai [1], Theorem 2.7, proves for the Coulomb potential $V(x)=(e/r)I$ that H is essentially self-adjoint if and only if $e^2\leqq m_0^2-(1/4)$.

Our result is that we can take $m=m_0$ in (2), that is,

Theorem 1. *If the potential $Q(x)$ satisfies*

(4) $$r\left|V(x)+\frac{i}{2r}\alpha_r\right|\leqq m_0,$$

then H is essentially self-adjoint.

Corollary 2. *Let $m_0\geqq(1/2)$.*

(i) *If $V(x)$ satisfies*

$$r|V(x)|\leqq m_0-(1/2),$$

then H is essentially self-adjoint.

(ii) *If $V(x)$ commutes with α_r and satisfies*

$$r^2|V(x)|^2\leqq m_0^2-(1/4),$$

then H is essentially self-adjoint.

The above assertion (ii) is a slight generalization of the "if" part in Arai [1], Theorem 2.7.

Corollary 3 (the case $b_1=b_2=0$). *If*

(i) *$V(x)$ satisfies $r|V(x)|\leqq(1/2)$,*

or

(ii) *$V(x)$ commutes with α_r and satisfies $r|V(x)|\leqq(\sqrt{3}/2)$, then*

$$H=\sum_{j=1}^{3}\alpha_j D_j+\beta+V(x)$$

is essentially self-adjoint.

The above (i) appears in Kato [2], Theorem 5.10, and (ii) in Yamada [3], § 4.

Proof of Theorem 1. The conditions (4) and (1) imply that $m_0\geqq(1/2)$, and that if $m_0=(1/2)$, then $V(x)\equiv 0$ (cf. Remark 9 for the detailed proof). If $m_0=(1/2)$, and therefore, $V(x)\equiv 0$, then our assertion follows from Arai [1], Theorem 2.7. Thus, we shall consider the case $m_0>(1/2)$. For the sake of technical treatments we put

$$V_R(x)=\chi_R(x)(1-ar+br^2)V(x), \tag{5}$$

where $\chi_R(x)$ is the characteristic function of $\{x;|x|\leqq R\}$ and

$$a=\frac{4}{4m_0^2-1},\qquad b=\frac{2}{4m_0^2-1}$$

(R will be determined later). Since the condition (4) implies

$$rV(x)\quad\text{is bounded in } \boldsymbol{R}^3, \tag{6}$$

$V(x)-V_R(x)$ is also bounded. Therefore we have only to prove the essential self-adjointness of

$$\tilde{H}=\sum_{j=1}^{3}\alpha_j D_j+\frac{ib_1}{r}\alpha_r\beta+\frac{b_2}{r}\beta+V_R(x)$$

on $[C_0^\infty(\boldsymbol{R}^3)]^4$. According to Kato [2] (Chap. V, § 3), the symmetric operator $\tilde{H}$ is essentially self-adjoint if and only if the range $R(\tilde{H}\pm i)$ is dense in $[L_2(\boldsymbol{R}^3)]^4$. Therefore, we complete the proof, if the following Lemma 4 is shown.

Lemma 4. *Let $\varepsilon=1$ or -1. Under the condition* (4) *there exist no non-trivial $[L_2(\boldsymbol{R}^3)]^4$-solutions satisfying*

$$\left(\sum_{j=1}^{3}\alpha_j D_j+\frac{ib_1}{r}\alpha_r\beta+\frac{b_2}{r}\beta+V_R\right)u=i\varepsilon u(x). \tag{7}$$

In order to prove Lemma 4 we shall prepare some propositions.

Proposition 5. *Let $u\in[L_2(\boldsymbol{R}^3)]^4$ satisfy* (7). *Then we have*

$$ru(x)\in[L_2(\boldsymbol{R}^3)]^4.$$

This proposition is obtained from (1), (6) and a fact that $u(x)$ satisfies

$$\left(\sum_{j=1}^{3} \alpha_j D_j - i\varepsilon\right)(ru) = -i\alpha_r u - ib_1\alpha_r\beta u - b_2\beta u - rV_R u \in [L_2(\boldsymbol{R}^3)]^4.$$

We set

$$S_\pm = \sum_{j=1}^{3} \alpha_j D_j + \frac{ib_1}{r}\alpha_r\beta + \frac{b_2}{r}\beta + \frac{i}{2r}\alpha_r \pm i.$$

Proposition 6. *Suppose that $u(x)$ is a solution of (7) belonging to $[L_2(\boldsymbol{R}^3)]^4$. Then we obtain*

$$\int_{\boldsymbol{R}^3} |S_\pm(ru)|^2\, dx \geqq \int_{\boldsymbol{R}^3} (m_0^2 - r + r^2)\,|u(x)|^2\, dx.$$

The above proposition follows from Proposition 5 and the proof of Lemma 3.4 in Arai [1].

Proposition 7. *If R is taken sufficiently small, we have*

$$\left|rV_R(x) + \frac{i}{2}\alpha_r\right|^2 \leqq m_0^2 - r + \frac{r^2}{2} \qquad (x \in \boldsymbol{R}^3).$$

Proof. Recall the definition (5) of $V_R(x)$. If R is sufficiently small,

$$0 \leqq \chi_R(x)(1 - ar + br^2) \leqq 1.$$

Then we have in consequence of (4) that

$$\begin{aligned}
&\left(rV_R + \frac{i}{2}\alpha_r\right)^*\left(rV_R + \frac{i}{2}\alpha_r\right)\\
&= \frac{1}{4} + \frac{ir}{2}(V_R\alpha_r - \alpha_r V_R) + r^2 V_R^2\\
&\leqq \frac{1}{4} + \left\{\frac{ir}{2}(V\alpha_r - \alpha_r V) + r^2V^2\right\}(1 - ar + br^2)\chi_R(x)\\
&= \frac{1}{4} + \left\{-\frac{1}{4} + \left(rV + \frac{i}{2}\alpha_r\right)^*\left(rV + \frac{i}{2}\alpha_r\right)\right\}(1 - ar + br^2)\chi_R(x)\\
&\leqq \frac{1}{4} + \left(m_0^2 - \frac{1}{4}\right)(1 - ar + br^2)\\
&= m_0^2 - a\left(m_0^2 - \frac{1}{4}\right)r + b\left(m_0^2 - \frac{1}{4}\right)r^2\\
&= m_0^2 - r + \frac{r^2}{2}.
\end{aligned}$$

Q.E.D.

Proof of Lemma 4. Now the proof of Lemma 4 is obvious. Let $u(x)$ be any $[L_2(\boldsymbol{R}^3)]^4$-solution of (7). We shall prove the case $\varepsilon = +1$ (the proof for $\varepsilon = -1$ is similarly obtained). Then we have from (7) and the definition of $S_\pm$ that

$$S_-(ru) = -\left(\frac{i}{2}\alpha_r + rV_R\right)u,$$

and, by virtue of Propositions 6 and 7,

$$\int_{\boldsymbol{R}^3} (m_0^2 - r + r^2)\,|u(x)|^2\, dx \leqq \int_{\boldsymbol{R}^3} \left(m_0^2 - r + \frac{r^2}{2}\right)|u(x)|^2\, dx,$$

which yields $u(x) = 0$. Q.E.D.

Remark 8. In Arai [1], Theorem 3.1, the essential self-adjointness of H is shown under the condition that for some constants $\tilde{m}$ and s

$$(8) \qquad r\left|V(x)-\frac{is}{r}\alpha_r\right| \leqq \tilde{m} < m_0+s-\frac{1}{2}, \qquad |s| \leqq \frac{1}{2}.$$

Recently, M. Arai points out in our private communication that it suffices to assume the case $s=(1/2)$ in (8), that is, the condition (8) implies the condition (2) with $m=\tilde{m}+(1/2)-s$ as follows

$$\begin{aligned} r\,|V(x)-(i/2r)\alpha_r| &= r\,|V(x)+(i/2r)\alpha_r| \\ &= r\,|V(x)+(is/r)\alpha_r+\{(i/2r)-(is/r)\}\alpha_r| \\ &\leqq r\,|V(x)+(is/r)\alpha_r|+(1/2)-s \\ &= r\,|V(x)-(is/r)\alpha_r|+(1/2)-s \\ &\leqq \tilde{m}+(1/2)-s=m \\ &\leqq m_0+s-(1/2)+(1/2)-s=m_0. \end{aligned}$$

Remark 9. Suppose that A and B are Hermitian symmetric. Then we have $\max(|A|, |B|) \leqq |A+iB|$. If the absolute value of every eigenvalue of B is equal to $|A+iB|$, then $A=0$.

In fact, the first assertion follows from

$$|A| = \sup_{|f|=1} |(Af, f)| \leqq \sup_{|f|=1} |(Af, f)+i(Bf, f)| \leqq |A+iB|,$$

and the similar estimate $|B| \leqq |A+iB|$. In order to prove the second assertion we shall take an arbitrary eigenvector f such that $|f|=1$ and $Bf=\lambda f$ ($|\lambda|=|A+iB|$). Noting that λ is a real number, we have

$$\begin{aligned} \lambda^2 = |A+iB|^2 &\geqq |(A+iB)f|^2 \\ &= |Af|^2+i(Bf, Af)-i(Af, Bf)+|Bf|^2 \\ &= |Af|^2+i\lambda(f, Af)-i\lambda(Af, f)+\lambda^2 \\ &= |Af|^2+\lambda^2, \end{aligned}$$

which yields $Af=0$. In view of the eigenvector expansion of the Hermitian symmetric matrix B we have $A=0$.

In conclusion, the author wishes to express his sincere gratitude to Prof. M. Arai for his enduring encouragement and valuable discussions.

References

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