113. On Triple L-functions

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(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 12, 1986)

We extend the Garrett's result [3] on triple products to different weight case. Details are described in [5]. Let n be a positive integer and $\Gamma_n = Sp(n, \boldsymbol{Z})$. We denote by H_n the Siegel upper half space of degree n. Let $S_k(\Gamma_1)$ be the space of cuspforms of weight k and of degree one. We write Fourier expansion of $f \in S_k(\Gamma_1)$ as $f(z) = \sum_{n=1}^{\infty} a(n, f) e^{2\pi i n z}$. If $f \in S_k(\Gamma_1)$ is a normalized Hecke eigenform and p is a prime, we define semi-simple $M_p(f) \in GL(2, \boldsymbol{C})$ (up to conjugate class) by $\det(1 - tM_p(f)) = 1 - a(p, f)t + p^{k-1}t^2$. For normalized Hecke eigenforms f, g and h, define 'triple L-function' $L(s; f, g, h)$ by

$$L(s; f, g, h) = \prod_{p:\text{prime}} \det(1 - p^{-s} M_p(f) \otimes M_p(g) \otimes M_p(h))^{-1}.$$

For Siegel modular forms $f_1, \cdots, f_m$ and a field K, we denote by $K(f_1, \cdots, f_m)$ the field generated by all the Fourier coefficients of $f_1, \cdots, f_m$ over K. If f and g are C^∞-modular forms (of degree one), we put

$$\langle f, g \rangle_k = \int_{\Gamma_1 \backslash H_1} f(x+iy)\overline{g(x+iy)} y^{k-2} dx dy$$

provided that it converges absolutely. For even integers $r \geq 0$, $k > 4$ and $f \in S_{k+r}(\Gamma_1)$, we denote by $[f]_r$ the Klingen type Eisenstein series attached to f and of type $\det^k \otimes \mathrm{Sym}^r\,\mathrm{St}$, which is a Siegel modular form of degree two. (Precise definition is given later.)

Theorem A. *Let k, l and m be even integers satisfying $k \geq l \geq m$ and $l + m - k > 4$. Let $f \in S_k(\Gamma_1)$, $g \in S_l(\Gamma_1)$ and $h \in S_m(\Gamma_1)$ be normalized Hecke eigenforms. Put*

$$\tilde{L}(s; f, g, h) = \Gamma_C(s)\Gamma_C(s-k+1)\Gamma_C(s-l+1)\Gamma_C(s-m+1)L(s; f, g, h)$$

where $\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s)$. Then $\tilde{L}(s; f, g, h)$ meromorphically extends to the whole s-plane and satisfies the functional equation

$$\tilde{L}(s; f, g, h) = -\tilde{L}(k+l+m-2-s; f, g, h).$$

Moreover, we have

$$(1) \qquad \pi^{5+k-3l-3m} L(l+m-2; f, g, h) / (\langle f, f \rangle_k \langle g, g \rangle_l \langle h, h \rangle_m) \in \boldsymbol{Q}([f]_{2k-l-m}, f, g, h)$$

and, if $L((k+l+m)/2-1; f, g, h)$ is finite,

$$L\left(\frac{k+l+m}{2} - 1; f, g, h\right) = 0.$$

Corollary. *Let $f \in S_k(\Gamma_1)$ be a normalized Hecke eigenform and $L_3(s, f)$ its third L-function. Put*

$$\tilde{L}_3(s, f) = \Gamma_C(s)\Gamma_C(s-k+1)L_3(s, f).$$

Then $\tilde{L}_3(s, f)$ satisfies the functional equation

$$\tilde{L}_3(s, f) = -\tilde{L}_3(3k-2-s, f).$$

Especially we have $L_3((3k/2)-1, f)=0$.

This functional equation coincides with the conjecture of Serre [6]. It is interesting that $L(s; f, g, h)$ and $L_3(s, f)$ *always* vanish at the center. Concerning special values at other points, we have the following result.

Theorem B. *Let f, g and h be normalized Hecke eigenforms of weight k. For an integer j with $0 \leq j \leq (k/2)-2$, we have*

$$\pi^{5-5k+4j} L(2k-2-j; f, g, h)/(\langle f, f\rangle_k \langle g, g\rangle_k \langle h, h\rangle_k) \in \mathbf{Q}(f, g, h).$$

We sketch the proof of Theorem A. The key idea different from Garret [3] is the use of vector valued Klingen type Eisenstein series of degree two in stead of Siegel's Eisenstein series of degree three. The corollary immediately follows from the relation of $L_3(s, f)$ and $L(s; f, f, f)$.

Let St be the standard representation of $GL(2, \mathbf{C})$. Let $q>4$ and $r \geq 0$ be even integers. We realize representation $\det^q \otimes \mathrm{Sym}^r\,\mathrm{St}$ on $\mathbf{C}^{r+1}$ as follows:

$$(\det^q \otimes \mathrm{Sym}^r\,\mathrm{St})\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix}_r = (ad-bc)^q \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right)_r$$

where $\begin{pmatrix} x \\ y \end{pmatrix}_r = {}^t(x^r, x^{r-1}y, \cdots, y^r)$. For an integer j with $0 \leq j \leq r$, we put $v_j = {}^t(0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 lies in the $(j+1)$-th column. This is compatible to the definition of v_0 in Arakawa [1, (0.2)]. We use $\{v_j \mid 0 \leq j \leq r\}$ as a base of the representation space $\mathbf{C}^{r+1}$. Set $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = a$. We define $\Gamma_{2,1}$ as the subgroup of Γ_2 consisting the elements whose entries in the first three columns and the last row are zero. For $s \in \mathbf{C}$ and $f \in S_{q+r}(\Gamma_1)$, we define (vector valued non-holomorphic) Klingen type Eisenstein series attached to f by

$$[f]_r(Z, s) = \sum_{M \in \Gamma_{2,1}\backslash\Gamma_2} \left(\frac{\det(\mathrm{Im}\, M\langle Z\rangle)}{\mathrm{Im}\, M\langle Z\rangle^*}\right)^s f(M\langle Z\rangle^*)(\det^q \otimes \mathrm{Sym}^r\,\mathrm{St})(J(M, Z)^{-1})v_0$$

where $M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ and $J(M, Z) = CZ+D$ for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and $Z \in H_n$. We put $[f]_r(Z) = [f]_r(Z, 0)$. By a method similar to Arakawa [1, Proposition 1.2], we see that $[f]_r(Z, s)$ converges absolutely if $\mathrm{Re}\,(q+2s) > 4$. Using a suitable differential operator, we generalize the result of Böcherer [2] as follows:

Proposition. *Let q, r and f be as above. Put*

$$K(s, f) = \pi^{-3s} 2^{-2s} \frac{\Gamma_3\left(s + \frac{q+r}{2}\right)\Gamma(s)\Gamma(2s-1)}{\Gamma_3(s)\left(s + \frac{q+r}{2} - 1\right)} L_2(2s-2+q+r, f)$$

and

$$\mathbf{E}_r(Z, s, f) = \prod_{j=1}^{r/2} (s-j) K(s+(q/2), f)[f]_r(Z, s)$$

where $\Gamma_3(s) = \prod_{j=0}^{2} \Gamma(s-(j/2))$ and $L_2(s, f)$ is the second L-function of f. Then, $[f]_r(Z, s)$ meromorphically extends to the whole s-plane and satisfies the functional equation

$$\boldsymbol{E}_r(Z, s, f)=\boldsymbol{E}_r(Z, 2-q-s, f).$$

Let $r=2k-l-m$. Let $F_{r,k-l}(z, w, s, f)$ be the v_{k-l}-component of the $[f]_r\left(\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}, s\right)$. After some computations, we have

$$(2) \quad \langle\langle F_{r,k-l}(z, w, s, f), g(z)\rangle_l, h(w)\rangle_m = 2(4\pi)^{2-s-l-m} \times \frac{\Gamma(s+l+m-2)\Gamma(s+l-1)\Gamma(s+m-1)}{\Gamma(2s+l+m-2)} \cdot \frac{L(s+l+m-2\,;\, f, g, h)}{L_2(2s+l+m-2, f)}.$$

Combining (2) with Proposition, we obtain Theorem A.

Remark. The form of (2) at $s=0$ gives affirmative support to the conjecture in [4, §4]. When $r=0$ or 2, it holds $\boldsymbol{Q}([f]_r)=\boldsymbol{Q}(f)$ (we use [4, Corollary 2.3] for $r=2$) and the value (1) is effectively computable.

Acknowledgement. The author would like to thank Prof. N. Kurokawa and Prof. S. Mizumoto for encouragement and suggestion.

References

[1] T. Arakawa: Vector valued Siegel's modular forms of degree two and the associated Andrianov L-functions. Manuscripta Math., **44**, 155–185 (1983).

[2] S. Böcherer: Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe. J. Reine Angew. Math., **362**, 146–168 (1985).

[3] P. B. Garrett: Decomposition of Eisenstein series: Rankin triple products (to appear in Ann. of Math.).

[4] T. Satoh: On certain vector valued Siegel modular forms of degree two. Math. Ann., **274**, 335–352 (1986).

[5] ——: Some remarks on triple L-functions (preprint) (1986).

[6] J.-P. Serre: Une interprétation des congruences relatives à la fonction τ de Ramanujan. Séminaire Delange-Pisot-Poitou, 1967–68, Exp. 14 (February 1968).