15. A Formulation of Noncommutative McMillan Theorem

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§1. **Introduction.** In this short note, we formulate and prove a McMillan type convergence theorem in a noncommutative dynamical system based on our works about entropy operators [1].

Before formulating the McMillan theorem, we discuss a description of noncommutative dynamical systems, a noncommutative message space and entropy operators.

A noncommutative dynamical system (NDS for short) can be described by a von Neumann algebraic triple or, more generally, a C^*-algebraic triple denoted by $(\mathfrak{N}, \mathfrak{S}, \alpha)$. Namely, $\mathfrak{N}$ is a von Neumann algebra or C^*-algebra, $\mathfrak{S}$ is the set of all states on $\mathfrak{N}$ and α is an automorphism of $\mathfrak{N}$ describing a certain evolution of the system. A self-adjoint element A of the algebra $\mathfrak{N}$ corresponds to a random variable in usual commutative dynamical (probability) systems (CDS for short) and a state in NDS corresponds to a probability measure in CDS. Here we use a von Neumann algebraic description for simplicity. Consult the bibliography [2] for NDS and noncommutative probability theory.

Let $\mathfrak{N}$ be a finite dimensional von Neumann (matrix) algebra acting on a Hilbert space $\mathcal{H}$ with a faithful normal tracial state τ, and let $P(\mathfrak{M})$ be the set of all *minimal finite partitions of unit I* in a von Neumann subalgebra $\mathfrak{M}$ of $\mathfrak{N}$. A set of projections $\tilde{P}=\{P_j\}$ is said to be a minimal partition of I in $\mathfrak{M}$ if $P_j \in \mathfrak{M}(\forall j)$, $P_i \perp P_j$ $(i \neq j)$ and $\sum P_j = I$ hold, and for each j there does not exist a projection E such as $0<E<P_j$. Since any two partitions $\tilde{P}=\{P_j\}$ and $\tilde{Q}=\{Q_j\}$ are unitary equivalent, *the entropy operator* $H_\tau(\mathfrak{M})$ and *the entropy* $S_\tau(\mathfrak{M})$ w.r.t. $\mathfrak{M}$ and τ can be uniquely defined as [1]:

(1.1) $$H_\tau(\mathfrak{M}) = -\sum_k P_k \log \tau(P_k)$$

(1.2) $$S_\tau(\mathfrak{M}) = \tau(H_\tau(\mathfrak{M}))$$

for any $\tilde{P}=\{P_j\} \in P(\mathfrak{M})$. The above entropy $S_\tau(\mathfrak{M})$ has already been discussed in [3, 4] without considering $H_\tau(\mathfrak{M})$.

Now for any von Neumann subalgebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathfrak{N}$ and any partition $\tilde{P}=\{P_j\} \in P(\mathfrak{M}_2)$, it is easily seen that $\tilde{P}$ is not always in $P(\mathfrak{M}_1 \vee \mathfrak{M}_2)$ but there exists a partition $\{P_{ij}\}$ in $P(\mathfrak{M}_1 \vee \mathfrak{M}_2)$ such that $P_j = \sum_i P_{ij}$, where

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$\mathfrak{M}_1 \vee \mathfrak{M}_2$ is the von Neumann algebra generated by $\mathfrak{M}_1$ and $\mathfrak{M}_2$. In the same vein as $H_\tau(\mathfrak{M})$ and $S_\tau(\mathfrak{M})$, we can formulate *the conditional entropy operator* and *the conditional entropy* by the following way [1]:

(1.3) $$H_\tau(\mathfrak{M}_1 | \mathfrak{M}_2) = -\sum P_{ij}\{\log \tau(P_{ij}) - \log \tau(P_j)\},$$

(1.4) $$S_\tau(\mathfrak{M}_1 | \mathfrak{M}_2) = \tau(H_\tau(\mathfrak{M}_1 | \mathfrak{M}_2)),$$

where $\{P_{ij}\} \in P(\mathfrak{M}_1 \vee \mathfrak{M}_2)$ and $\{P_j\} \in P(\mathfrak{M}_2)$ with $P_j = \sum_i P_{ij}$. This conditional entropy operator satisfies the similar relations carried by the conditional entropy function in CDS [5]. For example, we have

Theorem 1. (1) *$H_\tau(\mathfrak{M}) = H_\tau(\mathfrak{M} | CI)$, where $CI = \{\lambda I ; \lambda \in C\}$.*

(2) $$H_\tau(\mathfrak{M}_1 \vee \mathfrak{M}_2) = H_\tau(\mathfrak{M}_1 | \mathfrak{M}_2) + H_\tau(\mathfrak{M}_2).$$

(3) *If $\tau \circ \alpha = \alpha$, then $H_\tau(\alpha\mathfrak{M}_1 | \alpha\mathfrak{M}_2) = \alpha H_\tau(\mathfrak{M}_1 | \mathfrak{M}_2)$.*

This theorem has been proved in [1] and makes us easily handle the entropy operators in NDS.

Now we set a noncommutative message space, with respect to which we formulate the noncommutative McMillan theorem. Since τ is faithful normal, τ might be represented by a vector x in $\mathcal{H}$ such that

$$\tau(\cdot) = \langle x, \cdot x \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}$. Let $\mathcal{K}$ be the infinite tensor product of $\mathcal{H}$ with respect to the above vector x in the sense of von Neumann and be denoted by

$$\mathcal{K} = \otimes_{-\infty}^{\infty} \{\mathcal{H}, x\}.$$

We define the *noncommutative message space* $\mathcal{A}$ as the von Neumann algebra on $\mathcal{K}$ generated by operators $\overline{A}_n$ ($n \in Z$) defined as follows: for each $n \in Z$, let A_n be an operator on $\mathcal{H}$, then

$$\overline{A}_n(\otimes_{-\infty}^{\infty} x_k) = \otimes_{-\infty}^{\infty} x'_k \quad \text{with} \quad x'_k = \delta_{nk} A_n x_k + (1 - \delta_{nk}) x_k.$$

This message space is called the infinite tensor product of von Neumann algebra $\mathfrak{N}$ and is denoted by

$$\mathcal{A} = \otimes_{-\infty}^{\infty} \{\mathfrak{N}, \tau\}$$

where $\mathfrak{N}$ corresponds to the alphabet space in CDS. More general noncommutative message spaces and their physical properties will be discussed in terms of ergodic channels. Now, for the n-times tensor product Hilbert space $\mathcal{H}_n = \otimes^n \mathcal{H}$, every element Q in $B(\mathcal{H}_n)$, the set of all bounded linear operators on $\mathcal{H}_n$, can be canonically embedded into $B(\mathcal{H}_{n+1})$ in such a way that $Q \subsetneq Q \otimes I$, so that we have the canonical embedding j_n from $\mathfrak{N}_n = \otimes^n \mathfrak{N}$ (the n-times tensor product of von Neumann algebra $\mathfrak{N}$) into $\mathcal{A} = \otimes \mathfrak{N}$.

For a von Neumann subalgebra $\mathfrak{M}$ of $\mathfrak{N}$, let $\mathfrak{M}_n$ and $\mathcal{B}_n$ be

$$\mathfrak{M}_n = \otimes_1^n \mathfrak{M},$$

$$\mathcal{B}_n = j_n(\mathfrak{M}_n).$$

Then $\mathcal{B}_n$ becomes a von Neumann subalgebra (message subspace) of $\mathcal{A}$. Using a shift operator α defined as $\alpha(\otimes A_k) = \otimes A_{k+1}$ ($A_k \in \mathfrak{N}$ for each k), the above $\mathcal{B}_n$ is expressed by

$$\mathcal{B}_n = \bigvee_{k=0}^{n-1} \alpha^{-k} \mathcal{B}_1.$$

§ 2. McMillan convergence theorem. Our "information source" is now described by $(\mathcal{K}, \mathcal{A}, \alpha)$ and a state φ on the message space $\mathcal{A}$. The state φ controls the transmission of information, so that the McMillan theorem is written in terms of φ, α and the entropy operator defined by (1.1). We assume that φ_n, the restriction of φ to $\mathfrak{M}_n$, is tracial. Then the entropy operator w.r.t. φ and $\mathcal{B}_n$ is given by

(2.1) $$H_\varphi(\mathcal{B}_n) = -\sum_k Q_k^{(n)} \log \varphi_n(Q_k^{(n)}),$$

where $\{Q_k^{(n)}\}$ is a minimal finite partition of unit in $\mathfrak{M}_n$.

Theorem 2. *Under the above settings, we have* (1) *there exists an α-invariant operator h in $\mathcal{A}$ such that $H_\varphi(\mathcal{B}_n)/n$ converges to h φ-almost uniformly and strongly as $n \to \infty$*; (2) *if α is ergodic* (i.e., $\{A \in \mathcal{A}; \alpha(A) = A\} = CI$), *then $h = \varphi(h)I$.*

Proof. For any minimal finite partition $\{P_i; i=1, \cdots, N\}$ in $\mathfrak{M}$, the family $\{P_{i_1} \otimes P_{i_2} \otimes \cdots \otimes P_{i_n}\}$ is a minimal finite partition in $\mathfrak{M}_n$, where n indices $i_1, \cdots, i_n$ run from 1 to N. As $\mathfrak{M}_n$ is a finite dimensional von Neumann algebra, $H_\varphi(\mathcal{B}_n)$ defined in (2.1) can be expressed by

$$H_\varphi(\mathcal{B}_n) = -\sum_{i_1, \cdots, i_n=1}^{N} P_{i_1} \otimes \cdots \otimes P_{i_n} \log \varphi_n(P_{i_1} \otimes \cdots \otimes P_{i_n}).$$

Let us consider the von Neumann subalgebra $\mathcal{C}_n$ of $\mathfrak{M}_n$ generated by the family $\{P_{i_1} \otimes \cdots \otimes P_{i_n}\}: \mathcal{C}_n = \{P_{i_1} \otimes \cdots \otimes P_{i_n}; i_1, i_2, \cdots, i_n = 1, \cdots, N\}''$, and let $\mathcal{C}_1 = \{P_i\}'' = \mathcal{C}$. Then $\mathcal{C}_n$ is the n-times tensor product of $\mathcal{C}$, and $\mathcal{C}, \mathcal{C}_n$ are commutative von Neumann algebras, so that from Theorem 6.3 of [2], there exist compact Hausdorff spaces Ω_n, Ω and probability measures μ_n, μ such that

$$\mathcal{C}_n \cong L^\infty(\Omega_n, \mu_n),$$
$$\mathcal{C} \cong L^\infty(\Omega, \mu).$$

Moreover, $\mathcal{C}_n$ is monotone increasing and generates the infinite tensor product $\tilde{\mathcal{C}}$ of $\mathcal{C}$. Since $\tilde{\mathcal{C}}$ is commutative, there exist a compact Hausdorff space $\tilde{\Omega}$ and a probability measure $\tilde{\mu}$ such that

$$\tilde{\mathcal{C}} \cong L^\infty(\tilde{\Omega}, \tilde{\mu})$$

and

$$\varphi(P_{i_1} \otimes \cdots \otimes P_{i_n}) = \tilde{\mu}(\Delta_{i_1 \cdots i_n}),$$

where $P_{i_1} \otimes \cdots \otimes P_{i_n}$ corresponds to the characteristic function $1_{\Delta_{i_1 \cdots i_n}}$ for some measurable set $\Delta_{i_1 \cdots i_n}$ in $\prod_1^n \Omega$. Thus the commutative McMillan theorem together with Theorem 1 implies that our entropy operator $H_\varphi(\mathcal{B}_n)/n$ converges to some α-invariant operator $h \in \tilde{\mathcal{C}} = L^\infty(\tilde{\Omega}, \tilde{\mu})$ φ-a.u. because of φ-a.u. $= \tilde{\mu}$-a.e. Since $\mathcal{C}_n$ increasingly converges to $\tilde{\mathcal{C}}$ and there always exists the conditional expectation from $\tilde{\mathcal{C}}$ onto $\mathcal{C}_n$ [2, 6], the L^1-convergence of $H_\varphi(\mathcal{B}_n)/n$ is equal to the strong-convergence of $H_\varphi(\mathcal{B}_n)/n$ because of the Martingale convergence theorem of Umegaki [2, 7]. The α-invariance of h is trivial from the definition of $H_\varphi(\mathcal{B}_n)/n$.

(2) is an immediate consequence of (1). Q.E.D.

We would like to give a few remarks here: When we use a pure state (projection) as a quantum mechanical signal (alphabet) as usually done, the proof of the noncommutative McMillan theorem is essentially traced from that of the commutative McMillan theorem, which is a main claim of this paper. However when one tries to use a coherent state or a mixture of pure states as an input signal, the situation might be different, in which more general formulation of the entropy operators and the McMillan theorem would be desirable. This will be done by dropping some of our assumptions; for instance, (1) representing each element of message by a projection, (2) the finite dimensionality of $\mathfrak{N}$, (3) the traciality of $\varphi_n=\varphi \upharpoonright \mathfrak{M}_n$. Such a generalization with some applications in quantum information theorem [8, 10] and physical state change [9] will be discussed elsewhere.

References

[1] M. Ohya and A. Yoshikawa: Entropy operators and a convergence theorem in a noncommutative dynamical system (submitted).

[2] H. Umegaki, M. Ohya, and F. Hiai: Introduction of Operator Algebras. Kyoritsu Publishing Company (1985)(in Japanese).

[3] A. Connes and E. Størmer: Entropy for automorphisms of II_1 von Neumann algebras. Acta Math., **134**, 289–306 (1975).

[4] G. G. Emch: Positivity of the K-entropy on nonabelian K-flows. Z. Wahr. Gebiete, **29**, 241–252 (1974).

[5] H. Umegaki and M. Ohya: Entropies in Probabilistic Systems. Kyoritsu Publishing Company (1983) (in Japanese).

[6] H. Umegaki: Conditional expectation in an operator algebra. Tohoku Math. J., **6**, 177–181 (1954).

[7] ——: Conditional expectation in an operator algebra II. ibid., **8**, 86–100 (1956).

[8] M. Ohya: On compound state and mutual information in quantum information theory. IEEE Information Theory, **29**, 770–774 (1983).

[9] ——: State change and entropies in quantum dynamical systems. Springer Lect. Notes in Math., **1136**, 397–409 (1985).

[10] H. Umegaki and M. Ohya: Quantum Mechanical Entropies. Kyoritsu Publishing Company (1984) (in Japanese).