# 75. *Structural Operators for Linear Delay-differential Equations in Hilbert Space*

By Hiroki TANABE
Department of Mathematics, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1988)

Let $H$ and $V$ be complex Hilbert spaces such that $V$ is a dense subspace of $H$ and the inclusion mapping of $V$ into $H$ is continuous. The norms of $H$ and $V$ are denoted by $|\ |$ and $\|\ \|$ respectively. Identifying $H$ with its antidual we may write $V\subset H\subset V^*$. We use the notation $(\ ,\ )$ to denote both the innerproduct of $H$ and the pairing between $V^*$ and $V$. For a couple of Hilbert spaces $X$ and $Y$ the notation $B(X, Y)$ denotes the totality of bounded linear mappings of $X$ into $Y$, and $B(X)=B(X, X)$.

Let $a(u, v)$ be a sesquilinear form defined on $V\times V$. Suppose that there exist positive constants $C$ and $c$ such that

$$|a(u, v)|\leqq C\|u\|\|v\|, \qquad \operatorname{Re} a(u, u)\geqq c\|u\|^2$$

for any $u, v\in V$. Let $-A_0\in B(V, V^*)$ be the operator associated with this sesquilinear form: $(-A_0u, v)=a(u, v)$, $u, v\in V$. The realization of $A_0$ in $H$ which is the restriction of $A_0$ to $D(A_0)=\{u\in V: A_0u\in H\}$ is also denoted by the same letter $A_0$. It is known that $A_0$ generates an analytic semigroup in both $H$ and $V^*$.

Let $A_i$, $i=1,2$, be operators in $B(V, V^*)$. Then, $A_iA_0^{-1}\in B(V^*)$ for $i=1,2$. We assume that these two operators map $H$ to itself and $A_iA_0^{-1}\in B(H)$, $i=1,2$. We assume also that $A_i^*(A_0^*)^{-1}\in B(H)$, $i=1,2$, where $A_0^*, A_i^*\in B(V, V^*)$ are the adjoint operators of $A_0, A_i$.

Let $a(s)$ be a real valued Hölder continuous function in the interval $[-h, 0]$, where $h$ is some positive number. We consider the following delay-differential equation

$$du(t)/dt=A_0u(t)+A_1u(t-h)+\int_{-h}^0 a(s)A_2u(t+s)ds \tag{1}$$

which is considered as an equation in both $H$ and $V^*$. According to [3] the fundamental solution $W(t)$ of (1) can be constructed.

It is easily seen that the space

$$\left\{f\in V^*: \int_0^\infty \|A_0\exp(tA_0)f\|_*^2\,dt<\infty\right\}$$

coincides with $H$, where $\|\ \|_*$ is the norm of $V^*$. Hence, in view of [1] the semigroup $S(t)$ in $Z=H\times L^2(-h, 0; V)$ is defined by

$$S(t)g=(u(t; g), u(t+\cdot\,; g)), \qquad g=(g^0, g^1)\in Z$$

where $u(t; g)$ is the mild solution of (1) (cf. [2]) satisfying the initial condition

$$u(0; g)=g^0, \quad u(s; g)=g^1(s), \quad -h\leqq s<0.$$

Similarly, the semigroup $S_T(t)$ in the same space $Z$ is defined for the adjoint equation

$$dv(t)/dt = A_0^* v(t) + A_1^* v(t-h) + \int_{-h}^{0} a(s) A_2^* v(t+s)\, ds.$$

The first structural operator $F$ is defined by

$$Fg = ([Fg]^0, [Fg]^1), \qquad g = (g^0, g^1) \in Z,$$

$$[Fg]^0 = g^0, \qquad [Fg]^1 = A_1 g^1(-h-s) + \int_{-h}^{s} a(\tau) A_2 g^1(\tau - s) d\tau$$

(cf. [2]). As is easily seen $F \in B(Z, Z^*)$, where $Z^* = H \times L^2(-h, 0; V^*)$.

**Theorem 1.** $FS(t) = S_T^*(t)F$, $S^*(t)F^* = F^* S_T(t)$, $t \geqq 0$.

The above theorem can be established as Theorem 4.2 of [2] with the aid of Theorem 2 below. Let $G$ be the second structural operator (cf. [2]):

$$Gg = ([Gg]^0, [Gg]^1), \qquad g = (g^0, g^1) \in Z^*,$$

$$[Gg]^1(s) = W(s+h) g^0 + \int_{-h}^{0} W(s+h+\tau) g^1(\tau) d\tau, \qquad [Gg]^0 = [Gg]^1(0),$$

where $W(t)$ is the fundamental solution of (1). It is easy to see that $G \in B(Z^*, Z)$.

**Theorem 2.** $S(t)G = GS_T^*(t)$, $G^* S^*(t) = S_T(t) G^*$.

**Example.** Let $A_0$ be a strongly elliptic linear partial differential operator with the Dirichlet boundary condition, and $A_i$, $i = 1, 2$, be linear differential operators of the same order as $A_0$. Then, the hypothesis of the above theorems are satisfied if their coefficients are sufficiently smooth.

## References

[1] G. Di Blasio, K. Kunisch, and E. Sinestrary: Stability for abstract linear functional differential equations. Israel J. Math., **50**, 231–263 (1985).

[2] S. Nakagiri: Structural properties of functional differential equations in Banach spaces. Osaka J. Math. (to appear).

[3] H. Tanabe: On fundamental solution of differential equation with time delay in Banach space. Proc. Japan Acad., **64**A, 131–134 (1988).