9. *Configuration of Divisors and Reflexive Sheaves*

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0. Let X be a connected complex manifold of dimension $\geqq 2$ and X^1 a reduced but *reducible* divisor of X. In this note, by using the Cech-stratification theoretical method in [7], we construct sheaves in the title from X^1. A main property of such sheaves is:

(*) They have X^1 as their determinantal divisor (cf. § 1). We see that the two highly important bundles on the projective spaces, Horrocks-Mumford and null correlation bundles (cf. [4] and [6]) are constructed in the above manner. We also form some other interesting sheaves which seem to belong to new classes (cf. § 2). This note is a report of our recent works, cf. [8]. Details will appear elsewhere.

1. **Construction.** Set $X^2=\bigcup_{i\neq j}(X_i^1\cap X_j^1)$ and $\mathring{X}=X-X^2$, where X_i^1 runs through all irreducible components of X^1. Then our sheaf, denoted by $\mathcal{E}$, is obtained as the direct image $i_*\mathring{\mathcal{E}}$ of a bundle $\mathring{\mathcal{E}}$ over $\mathring{X}$, with the injection $i:\mathring{X}\hookrightarrow X$. In order to form $\mathring{\mathcal{E}}$ we take (1) two open subsets N_0, N_1 of $\mathring{X}$ satisfying $N_0\cup N_1=\mathring{X}$ and (2) a non singular matrix $H\in \mathrm{GL}_r(\Gamma(N_0\cap N_1, \mathcal{O}))$, where $r=\operatorname{rank}\mathring{\mathcal{E}}$ and $\mathcal{O}$=structure sheaf of X. Then the bundle $\mathring{\mathcal{E}}$ is the one determined by H. We choose N_0, N_1 and H in such a manner that properties of X^1 reflect closely to them. More precisely we impose the following condition on $N_0, \cdots$:

(1.1) $N_0=X-X^1$ and $N_1=\amalg_{i\in\Delta_m}N_{1,i}$, where $N_{1,i}$ is an open neighborhood of $\mathring{X}_i^1:=X_i^1-X^2$ in $\mathring{X}$. (Here $\Delta_m=\{1, \cdots, m\}$ with m=the number of the irreducible components of X^1.)

(1.2) $H\in M_r(\Gamma(N_1, \mathcal{O}))$ and $(\det H)_0=\mathring{X}^1(:=\amalg_{i\in\Delta_m}\mathring{X}_i^1)$.

We see immediately that there are frames e^i of $\mathcal{E}_{|N_i}$, $i=0,1$, satisfying $e^0=e^1H$ in $N_0\cap N_1$. This implies that $e^0\subset\Gamma(X, \mathcal{E})$ and that X^1 is the determinantal divisor of $\mathcal{E}$: $(\det e^0)_{|\mathring{X}}=\mathring{X}^1$.

2. **Examples.** Here we assume that $\dim X\geqq 3$. Also we assume that (1) there is a line bundle $\mathcal{L}$ over X and sections $s_i\in\Gamma(\mathcal{L})$ such that $X_i^1=(s_i)_0$ and (2) for each $I=\{i_1, \cdots, i_s\}$ satisfying $X_I:=\bigcap_{i\in I}X_i^1\neq\phi$, $\operatorname{codim}_X X_I=s$ and X_I is smooth.

2.1. First we consider two types of matrices $H\in M_2(\Gamma(N_1, \mathcal{O}))$ as follows. (In (2.1, 2) below, $i\in\Delta_m$.)

(2.1) $$H_{|N_{1,i}}=\begin{bmatrix}1 & t_i\otimes s_i/\prod_{j\in\Delta_m}s_j\\ 0 & s_i/s_{i+1}\end{bmatrix}$$ where t_i is an element of $\Gamma(\mathcal{L}^{\otimes(m-1)})$,

and

(2.2) $$H_{|N_{1,i}}=\begin{bmatrix}1 & \prod_{\alpha=n+1}^{2n} s_{i\pm\alpha}/\prod_{\alpha=1}^{n} s_{i\pm\alpha}\\ 0 & s_i/s_{i+1}\end{bmatrix}.$$

For the sheaf $\mathcal{E}$, which is determined by H, we have:

Theorem 1. *In the both cases* (2.1) *and* (2.2), *the sheaves $\mathcal{E}$ are simple, i.e., $\Gamma(\mathcal{E}nd\,\mathcal{E})\simeq \boldsymbol{C}$, and so are indecomposable. Moreover,*

(2.3) $\operatorname{codim}_X S(\mathcal{E})\geqq 4$, *where* $S(\mathcal{E})=\{p\in X\,;\,\mathcal{E}_p$ *is not* $\mathcal{O}_p$*-free*$\}$.

Furthermore, if $m=5$ *in* (2.2) *then* $\operatorname{codim}_X S(\mathcal{E})=5$.

A bundle $\mathcal{F}$ over X may be said to be *of low rank* (*with respect to* $\dim X$), if $\operatorname{rank}\mathcal{F}<\dim X$. By generalizing this, we may say that a reflexive sheaf $\mathcal{F}$ is of low rank, if $\operatorname{codim}_X S(\mathcal{F})\geqq \operatorname{rank}\mathcal{F}+2$. (Note that, if $\operatorname{rank}\mathcal{F}=\dim X-1$, then the above condition implies that $\mathcal{F}$ is locally free and of low rank.) It is hard to find indecomposable reflexive sheaves of low rank (cf. [6]). Theorem 1 gives examples of such sheaves and contains more. For example, assume that $X=\boldsymbol{P}_n$, $n\geqq 3$, and $\mathcal{L}=\mathcal{O}_X(1)$ (=hyperplane bundle). Let X^1 be the union of m hyperplanes of X, which are linearly independent, $2\leqq m\leqq n$.

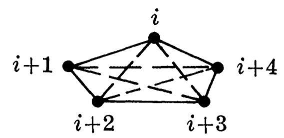
Corollary. *There is a simple reflexive sheaf $\mathcal{E}$ over X of rank two such that* (1) *$\mathcal{E}$ is simple,* (2) $\operatorname{codim}_X S(\mathcal{E})\geqq 4$ *and* (3) *X^1 is a determinantal divisor of $\mathcal{E}$.*

Remark. In (2.1, 2) assume that $(X,\mathcal{L},m)=(\boldsymbol{P}_3,\mathcal{O}_X(1),2)$ and $=(\boldsymbol{P}_4,\mathcal{O}_X(1),5)$. Moreover, in the first case, assume that s_1, s_2, t_1, t_2 are linearly independent elements of $\Gamma(\mathcal{O}_X(1))$. Then the (1, 2)-component of the matrix H takes the following simple form:

(2.4) t_i/s_i in the case of $\boldsymbol{P}_3$, and $s_{i+2}s_{i+3}/s_{i+1}s_{i+4}$ in the case of $\boldsymbol{P}_4$.

By Theorem 1 the sheaf $\mathcal{E}$, which is constructed from H, is locally free. Moreover, we have:

(2.5) $\mathcal{E}$=null correlation bundle in the case of $\boldsymbol{P}_3$, and =Horrocks-Mumford bundle in the case of $\boldsymbol{P}_4$.



The first one seems to be the oldest example of indecomposable bundles on $\boldsymbol{P}_n$ of low rank, $n\geqq 3$, and the second seems to be the most famous one among bundles on $\boldsymbol{P}_n$ of low rank, $n\geqq 4$. (Note that, in the (1, 2)-component for the second one (cf. (2.4)) the pentagon phenomenon appears; see [1] and the figure just above.)

2.2. Here we give two examples of matrices $H\in M_r(\Gamma(N_1,\mathcal{O}))$, $r\geqq 3$.

(2.6) $$H_{|N_{1,i}}=\begin{bmatrix}I_{r-1} & u_\alpha\otimes s_1/\prod_{j\in\Delta}s_j\\ 0 & s_i/s_{i+1}\end{bmatrix},\ \text{with } u_\alpha\in\Gamma(\mathcal{L}^{\otimes(m-1)})\,;\ 1\leqq\alpha\leqq r-1,$$

and

(2.7) the (α,β)-component of $H_{|N_{1,i}}=s_\alpha^{\otimes(m-1)}$; $\alpha\neq\beta$, and $=\coprod_{j\in\Delta_m}s_j/s_i$.

The matrix H defines a sheaf $\mathcal{E}$ over X.

Theorem 2. *In the both cases* (2.6, 2.7), *$\mathcal{E}$ is simple. Moreover, the singular locus $S(\mathcal{E})$ of $\mathcal{E}$ is as follows:*

$$S(\mathcal{E})=X^2\cap\left(\textstyle\bigcap_{\alpha=1}^{r-1}(u_\alpha)_0\right) \textit{ in the case of } (2.6), \textit{ and}$$
$$=\textstyle\bigcap_{i=1}^{m}(s_i)_0 \textit{ in the case of } (2.7).$$

Thus, in general, $\operatorname{codim}_X S(\mathcal{E})=r+1$ or $=m$. Letting the configuration of hyperplanes X^1 of the projective space $\boldsymbol{P}_n$, $n\geqq 3$, be as in Corollary to Theorem 1, we have:

Corollary. *There is a simple bundle $\mathcal{E}$ over X of* rank n ($=\dim \boldsymbol{P}_n$), *which has X^1 as its determinantal divisor.*

This follows from the first example. The second gives an example of a reflexive sheaf over $\boldsymbol{P}_n$ of rank n, which is singular at a single point. (The singularity of the determinantal divisor of a general section $f\in \Gamma^r(\mathcal{E})$ is toric: $s_\alpha^{\otimes(m-1)}+(\prod_{\beta\in\Delta_m}s_\beta)/s_\alpha=0$, with some $\alpha\in\Delta_m$.)

Remark. Treatments of our bundles on algebraic surfaces are done by Hino ([2]). He shows that, on $\boldsymbol{P}_2$, each stable bundle of rank two is, after tensoring with $\mathcal{O}_{\boldsymbol{P}_2}(m)$, $m\in \boldsymbol{Z}$, a deformation of a bundle of our type.

The proof of Theorems 1 and 2 depends on (1) investigations of local structure of the direct image sheaf $\mathcal{E}$ and (2) those of $\mathcal{E}nd\,\mathcal{E}$. The second is a refinement of § 4 of [7] and is done by analyzing the adjoint of H. The first is a central subject of the sheaves of the type of the present paper, and is done inductively on $X^1\supset X^2\supset X^3:=\bigcap_{i,j,k}(X_i^1\cap X_j^1\cap X_k^1)\supset\cdots$, by developping some general cohomological arguments. Details are in [8], and we want to give a refined version elsewhere.

3. Some remarks. The arguments hitherto may show that theory of configuration of divisors naturally enters into theory of vector bundles and reflexive sheaves, while the former has appeared in interesting geometric problems, e.g., theory of branched coverings (cf. [3] and [5]) and topology of complement of a union of divisors (cf. [6]). The content of the present note seems to deserve to be studied further. In this direction we propose two problems.

(The projective spaces $\boldsymbol{P}_n$ and the union X^1 of hyperplanes are as in § 2. In the first problem we assume that $n\geqq 4$.)

Problem. Determine if there are indecomposable reflexive sheaves on $\boldsymbol{P}_n$, which satisfy the following condition.

(3) $2\leqq \operatorname{rank}\mathcal{E}\leqq n-1$, and $\mathcal{E}$ has X^1 as its determinantal divisor.

We see that null correlation bundles give an affirmative answer for the case: $n=$ an odd integer $\geqq 5$, $m=n-1$, and $\operatorname{rank}\mathcal{E}=n-1$. But, at the present moment, we can not construct new examples for the above problem. The second problem concerns a generalization of the aforementioned result of Hino.

Problem. Determine what value $c=(c_1,\cdots,c_n)$, $n\in\boldsymbol{Z}$, can be represented by the Chern classes $(c_1(\mathcal{E}),c_2(\mathcal{E}),\cdots,c_n(\mathcal{E}))$ of an indecomposable bundle of rank n over $\boldsymbol{P}_n$, which is of the type considered in § 1.

In the above we identify $H^{2i}(\boldsymbol{P}_n,\boldsymbol{Z})$ with $\boldsymbol{Z}$, $0\leqq i\leqq n$.

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