92. *A Property of Certain Analytic Functions Involving Ruscheweyh Derivatives*

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1. Introduction. Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty} a_k z^k \qquad (p\in\mathcal{N}=\{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z: |z|<1\}$. For functions $f_j(z)$ ($j=1, 2$) defined by

$$f_j(z)=z^p+\sum_{k=p+1}^{\infty} a_{k,j} z^k, \tag{1.2}$$

we define the convolution $f_1*f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1*f_2(z)=z^p+\sum_{k=p+1}^{\infty} a_{k,1}a_{k,2} z^k. \tag{1.3}$$

With the convolution above, we define

$$D^{n+p-1}f(z)=\left(\frac{z^p}{(1-z)^{n+p}}\right)*f(z) \qquad (f(z)\in\mathcal{A}(p)), \tag{1.4}$$

where n is any integer greater than $-p$. We note that

$$D^{n+p-1}f(z)=\frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!}. \tag{1.5}$$

The symbol D^{n+p-1} when $p=1$ was introduced by Ruscheweyh [5], and the symbol D^{n+p-1} was introduced by Goel and Sohi [3]. Therefore, one called the sympol D^{n+p-1} the Ruscheweyh derivative of $(n+p-1)$th order. It follows from (1.5) that

$$z(D^{n+p-1}f(z))'=(n+p)D^{n+p}f(z)-nD^{n+p-1}f(z). \tag{1.6}$$

Recently, Chen and Lan ([1], [2]) have proved some interesting results of certain analytic functions involving Ruscheweyh derivatives.

2. A property. In order to derive our main result, we need the following lemma due to Miller and Mocanu [4].

Lemma. *Let $\phi(u, v)$ be a complex valued function,*

$$\phi: \mathcal{D}\longrightarrow C,\ \mathcal{D}\subset C^2 \qquad (C \text{ is the complex plane}),$$

and let $u=u_1+iu_2$, $v=v_1+iv_2$. Suppose that the function $\phi(u, v)$ satisfies

(i) *$\phi(u, v)$ is continuous in $\mathcal{D}$;*

(ii) *$(1, 0)\in\mathcal{D}$ and* $\mathrm{Re}\{\phi(1, 0)\}>0$;

(iii) *for all $(iu_2, v_1)\in\mathcal{D}$ such that $v_1\leq(-1+u_2^2)/2$,* $\mathrm{Re}\{\phi(iu_2, v_1)\}\leq 0$.

Let $q(z)=1+q_1z+q_2z^2+\cdots$ be regular in $\mathcal{U}$ such that $(q(z), zq'(z))\in\mathcal{D}$ for all $z\in\mathcal{U}$. If

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$$\mathrm{Re}\{\phi(q(z), zq'(z))\}>0 \qquad (z\in \mathcal{U}),$$

then $\mathrm{Re}\{q(z)\}>0$ $(z\in\mathcal{U})$.

Applying the above lemma, we prove

Theorem. *If a function* $f(z)\in\mathcal{A}(p)$ *satisfies*

$$\mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\}>\alpha \qquad (z\in\mathcal{U}) \tag{2.1}$$

for some α $(0\leq\alpha<1)$, *then*

$$\mathrm{Re}\left[\frac{D^{n+p-1}f(z)}{z^p}\right]^{\beta}>\gamma \qquad (z\in\mathcal{U}), \tag{2.2}$$

where

$$0<\beta\leq\frac{1}{2(n+p)(1-\alpha)} \tag{2.3}$$

and

$$\gamma=\frac{1}{2\beta(n+p)(1-\alpha)+1}. \tag{2.4}$$

Proof. Defining the function $q(z)$ by

$$\left[\frac{D^{n+p-1}f(z)}{z^p}\right]^{\beta}=\gamma+(1-\gamma)q(z), \tag{2.5}$$

we see that $q(z)=1+q_1z+q_2z^2+\cdots$ is regular in $\mathcal{U}$. Taking the logarithmic differentiations of both sides in (2.5), we have

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}=p+\frac{(1-\gamma)zq'(z)}{\beta(\gamma+(1-\gamma)q(z))}. \tag{2.6}$$

Using (1.6), (2.6) implies

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\alpha=1-\alpha+\frac{(1-\gamma)zq'(z)}{\beta(n+p)(\gamma+(1-\gamma)q(z))}, \tag{2.7}$$

or

$$\mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\alpha\right\}=\mathrm{Re}\left\{1-\alpha+\frac{(1-\gamma)zq'(z)}{\beta(n+p)(\gamma+(1-\gamma)q(z))}\right\}>0. \tag{2.8}$$

Letting

$$\phi(u, v)=1-\alpha+\frac{(1-\gamma)v}{\beta(n+p)(\gamma+(1-\gamma)u)}, \tag{2.9}$$

we see that

(i) $\phi(u, v)$ is continuous in $\mathcal{D}=(\mathcal{C}-\{\gamma/(\gamma-1)\})\times\mathcal{C}$;

(ii) $(1, 0)\in\mathcal{D}$ and $\mathrm{Re}\{\phi(1, 0)\}=1-\alpha>0$;

(iii) for all $(iu_2, v_1)\in\mathcal{D}$ such that $v_1\leq-(1+u_2^2)/2$,

$$\begin{aligned}\mathrm{Re}\{\phi(iu_2, v_1)\}&=1-\alpha+\frac{\gamma(1-\gamma)v_1}{\beta(n+p)(\gamma^2+(1-\gamma)^2u_2^2)}\\&\leq1-\alpha-\frac{\gamma(1-\gamma)(1+u_2^2)}{2\beta(n+p)(\gamma^2+(1-\gamma)^2u_2^2)}\\&=\frac{\{2\beta(n+p)(1-\alpha)-1\}u_2^2}{2\beta(n+p)(\gamma^2+(1-\gamma)^2u_2^2)}\leq0\end{aligned} \tag{2.10}$$

because $0<\beta\leq 1/2(n+p)(1-\alpha)$, $0\leq\alpha<1$ and $n>-p$. Thus the function $\phi(u, v)$ satisfies the conditions in Lemma. Therefore, applying Lemma, we have $\mathrm{Re}\{q(z)\}>0$ $(z\in\mathcal{U})$, that is,

$$\text{(2.11)}\qquad \mathrm{Re}\left[\frac{D^{n+p-1}f(z)}{z^p}\right]^{\beta}>\gamma=\frac{1}{2\beta(n+p)(1-\alpha)+1}.$$

This completes the proof of our theorem.

Taking $\beta=1/2$ in Theorem, we have

Corollary 1. *If $f(z)\in\mathcal{A}(p)$ satisfies the condition* (2.1),

$$\text{(2.12)}\qquad \mathrm{Re}\sqrt{\frac{D^{n+p-1}f(z)}{z^p}}>\frac{1}{(n+p)(1-\alpha)+1}\qquad (z\in\mathcal{U}),$$

where $1-1/(n+p)\leq\alpha<1$.

Letting $\beta=1/2(n+p)(1-\alpha)$, Theorem leads to

Corollary 2. *If $f(z)\in\mathcal{A}(p)$ satisfies the condition* (2.1), *then*

$$\text{(2.13)}\qquad \mathrm{Re}\left[\frac{D^{n+p-1}f(z)}{z^p}\right]^{\frac{1}{2(n+p)(1-\alpha)}}>\frac{1}{2}\qquad (z\in\mathcal{U}).$$

Putting $p=1$ and $n=1$ in Theorem, we have

Corollary 3. *If $f(z)\in\mathcal{A}(1)$ satisfies*

$$\text{(2.14)}\qquad \mathrm{Re}\left[\frac{zf''(z)}{f'(z)}\right]>2\alpha-1\qquad (z\in\mathcal{U}),$$

then

$$\text{(2.15)}\qquad \mathrm{Re}\,(f'(z))^{\beta}>\frac{1}{4\beta(1-\alpha)+1}\qquad (z\in\mathcal{U}),$$

where $0<\beta\leq(1-\alpha)/4$.

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