3. Remarks on the Stability of Certain Periodic Solutions of the Heat Convection Equations

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§1. **Introduction.** Let $\Omega(t)$ be a time-dependent bounded space domain in R^m ($m=2$ or 3) whose boundary $\partial\Omega(t)$ consists of two components, namely, $\partial\Omega(t)=\Gamma_0\cup\Gamma(t)$. Here Γ_0 is the inner boundary and $\Gamma(t)$ is the outer one. Moreover, these two boundaries do not intersect each other. We denote by K the compact set which is bounded by Γ_0. Let $u=u(x,t)$, $\theta=\theta(x,t)$ and $p=p(x,t)$ be the velocity of the viscous fluid, the temperature and the pressure, respectively. We consider the heat convection equation (HC) of Boussinesq approximation in $\hat{\Omega}=\bigcup_{0<t<T}\Omega(t)\times\{t\}$ with boundary conditions

(1) $\qquad u|_{\partial\Omega(t)}=\beta(x,t),\quad \theta|_{\Gamma_0}=T_0>0,\quad \theta|_{\Gamma(t)}=0$ for any $t\in(0,T)$.

In our previous paper [4], we have proven the unique existence of the time-periodic strong solution of (HC) with (1), provided the domain $\Omega(t)$ and the boundary data $\beta(x,t)$ both vary periodically with period T. The purpose of this paper is to show the asymptotic stability of the periodic solution which is obtained in [4].

§2. **Assumptions and results.** We make some assumptions:

(A1) For any fixed $t>0$, $\Gamma(t)$ and Γ_0 are both simple closed curves (or surfaces) and also they are of class C^3.

(A2) $\Gamma(t)\times\{t\}$ $(0<t<T)$ changes smoothly (say, of class C^4) with respect to t. (See, Assumptions II and III in [4].)

(A3) $g(x)$ is a bounded and continuous vector function in $R^m\backslash\text{int}\,K$.

(A4) $\beta(x,t)$ is sufficiently smooth in x and t. Moreover, it satisfies the following condition

$$\int_{\partial\Omega(t)}\beta\cdot n\,dS=0,$$

where n is the outer normal vector to $\partial\Omega(t)$.

(A5) The domain $\Omega(t)$ and the function $\beta(x,t)$ vary periodically in t with period $T>0$, i.e., $\Omega(t+T)=\Omega(t)$, $\beta(\cdot,t+T)=\beta(\cdot,t)$ for each $t>0$.

Since $\Omega(t)$ is bounded, there exists an open ball B_1 with radius d such that $\overline{\Omega(t)}\subset B_1$. We put $B=B_1\backslash K$. We introduce a solenoidal periodic function b over B such that $b(x,t)=\beta(x,t)$ on $\partial\Omega(t)$ and an appropriate function $\bar{\theta}$ on $\Omega(t)$ with the same boundary values on $\partial\Omega(t)$ as θ.

We now set the periodicity condition

(2) $\qquad u(\cdot,0)=u(\cdot,T)\qquad$ in $\Omega(0)=\Omega(T)$,

and consider the periodic problem for (HC) with (1) and (2).

Then we have in [4] the following theorem:

Theorem A. *In addition to assumptions* (A1)–(A5), *if the viscocity* ν *is sufficiently large and the boundary data* β *and* T_0 *are sufficiently small in some sense, then the periodic problem for* (*HC*) *has a unique strong solution with period* T.

Remark 1. The definition of strong solutions is to be given in §3. Detailed conditions on ν, β and T_0 in the above theorem are contained in [4].

We have now the following stability theorem which is the main result in this paper. (Symbols $W_2^p(\Omega)$, $\mathring{W}_2^p(\Omega)$ and $H_\sigma^1(\Omega)$ are used as usual.)

Theorem B. *Let* $W(t)={}^t(w(t),\psi(t))$ *be the periodic strong solution in Theorem A and let* $U_0={}^t(u_0,\theta_0)\in H_\sigma^1(\Omega(0))\times\mathring{W}_2^1(\Omega(0))$. *Then there exist positive numbers* ν_* *and* γ_* *independent of* $T\geqq 1$ *such that if* $\nu>\nu_*$,

$$\sup_{0\leqq t\leqq T}\|\nabla\bar{\theta}(t)\|_{L^2_{(\Omega(t))}}<\gamma_*,\quad \sup_{0\leqq t\leqq T-1}\left(\int_t^{t+1}\|b(s)\|^2_{W_2^2(B)}ds\right)^{1/2}<\gamma_*,$$

$$\sup_{0\leqq t\leqq T-1}\left(\int_t^{t+1}\|b_s(s)\|^2_{L^2(B)}ds\right)^{1/2}<\gamma_*,\quad \sup_{0\leqq t\leqq T}\|b(t)\|_{W_2^1(B)}<\gamma_*$$

$$\textit{and}\quad \|U_0\|_{\{W_2^1(\Omega(0))\}^{m+1}}<\gamma_*,$$

then the followings hold:

(i) *The initial value problem for* (*HC*) *with* (1) *and*

(3) $$u(0)=w(0)+u_0,\quad \theta(0)=\psi(0)+\theta_0 \qquad \textit{in } \Omega(0)$$

has a unique global strong solution.

(ii) *Let us denote the global strong solution obtained in* (i) *by* $V(t)={}^t(v(t),\Theta(t))$, *then we have*

(4) $$\|V(t)-W(t)\|_{\{L^2(\Omega(t))\}^{m+1}}\longrightarrow 0 \qquad \textit{as } t\to\infty.$$

§3. Strong solutions of the heat convection equation. We make a suitable change of variables and use the same letters after changing of variables, then (HC) and (1) are transformed to the followings:

(5) $$\begin{cases} u_t+(u\cdot\nabla)u=-\nabla p-(u\cdot\nabla)b-(b\cdot\nabla)u-R\theta+\Delta u+f_1 & \text{in } \hat{\Omega}, \\ \operatorname{div} u=0 & \text{in } \hat{\Omega}, \\ \theta_t+(u\cdot\nabla)\theta=\dfrac{1}{P}\Delta\theta-(u\cdot\nabla)\bar{\theta}-(b\cdot\nabla)\theta+f_2 & \text{in } \hat{\Omega}, \end{cases}$$

(6) $$u|_{\partial\Omega(t)}=0,\quad \theta|_{\partial\Omega(t)}=0 \qquad \text{for any } t\in(0,T),$$

where $f_1=-b_t-(b\cdot\nabla)b+\Delta b+d^3g/\nu^2-R(\bar{\theta}-1/P)$, $f_2=-(b\cdot\nabla)\bar{\theta}$, $R=\alpha gT_0d^3/\kappa\nu$, $P=\nu/\kappa$; ν, κ, α, ρ are physical constants and $g=g(x)$ is the gravitational vector.

Let us put $U={}^t(u,\theta)$ and we notice $H_\sigma(B)\times L^2(B)=(H_\sigma(B)\times 0)+(0\times L^2(B))$ (direct sum). Then we introduce a proper lower semicontinuous convex (p.l.s.c.) function as follows:

(7) $$\varphi_B(U)=\begin{cases} \dfrac{1}{2}\displaystyle\int_B\left(|\nabla u|^2+\frac{1}{P}|\nabla\theta|^2\right)dx & \text{if } U\in H_\sigma^1(B)\times\mathring{W}_2^1(B), \\ +\infty & \text{if } U\in(H_\sigma(B)\times L^2(B))\backslash(H_\sigma^1(B)\times\mathring{W}_2^1(B)). \end{cases}$$

Next we consider a closed convex set $K(t)$ of $H_\sigma(B)\times L^2(B)$:

(8) $$K(t)=\{U\in H_\sigma(B)\times L^2(B);\ U=0 \text{ a.e. in } B\backslash\Omega(t)\}$$

for any $t\in[0,T]$ and denote its indicator function by $I_{K(t)}$, namely, $I_{K(t)}(U)$

$=0$ if $U\in K(t)$ and $I_{K(t)}(U)=+\infty$ if $U\in(H_\sigma(B)\times L^2(B))\backslash K(t)$. Then we define the following p.l.s.c. function

(9) $$\varphi^t(U)=\varphi_B(U)+I_{K(t)}(U) \qquad \text{for every } t\in[0,T].$$

Let $\partial\varphi^t$ be the subdifferential operator of φ^t, then we see:

(i) $D(\partial\varphi^t)=\{U\in H_\sigma(B)\times L^2(B);\ U|_{\Omega(t)}\in(W_2^2(\Omega(t))\cap H_\sigma^1(\Omega(t)))$
$\times(W_2^2(\Omega(t))\cap \mathring{W}_2^1(\Omega(t))),\ U|_{B\backslash\Omega(t)}=0\}$

(ii) $\partial\varphi^t(U)=\{f\in H_\sigma(B)\times L^2(B);\ P(\Omega(t))f|_{\Omega(t)}=A(\Omega(t))U|_{\Omega(t)}\}$,

where $A(\Omega(t))$ is the Stokes operator, $P(\Omega(t))={}^t(P_\sigma(\Omega(t)),\ 1_{\Omega(t)})$ and $P_\sigma(\Omega(t))$ is the orthogonal projection from $L^2(\Omega(t))$ to $H_\sigma(\Omega(t))$. Using the operator $\partial\varphi^t$, we reduce (5) and (6) to the following abstract heat convection equation (AHC) in $H_\sigma(B)\times L^2(B)$.

(AHC) $$\frac{dV}{dt}+\partial\varphi^t(V(t))+F(t)V(t)+M(t)V(t)\ni P(B)\tilde{f}(t),\ t\in[0,T],$$

where $V={}^t(v,\theta)$, $F(t)V(t)={}^t(P_\sigma(B)(v\cdot\nabla)v,\ (v\cdot\nabla)\theta)$, $M(t)V(t)={}^t(P_\sigma(B)((v\cdot\nabla)b+(b\cdot\nabla)v+R\theta),\ (v\cdot\nabla)\tilde{\bar{\theta}}+(b\cdot\nabla)\theta)$, $\tilde{f}={}^t(\tilde{f}_1,\tilde{f}_2)$ and $P(B)={}^t(P_\sigma(B),1_B)$; $\tilde{f}_i$ means the natural extension of f_i.

Now we define the strong solution of (AHC) as follows.

Definition 1. Let $V:[0,S]\to H_\sigma(B)\times L^2(B)$, $S\in(0,T]$. Then V is called a strong solution of (AHC) on $[0,S]$ if it satisfies the following properties (i) and (ii).

(i) $V\in C([0,S];H_\sigma(B)\times L^2(B))$ and $dV/dt\in L^2(0,S;H_\sigma(B)\times L^2(B))$.

(ii) $V(t)\in D(\partial\varphi^t)$ for a.e. $t\in[0,S]$ and there is a function $G\in L^2(0,S;H_\sigma(B)\times L^2(B))$ satisfying $G(t)\in\partial\varphi^t(V(t))$ and $(dV/dt)+G(t)+F(t)V(t)+M(t)V(t)=P(B)\tilde{f}(t)$ for a.e. $t\in[0,S]$.

Definition 2. A strong solution of (AHC) satisfying the following condition (10) (resp. (11)) is called a periodic strong solution (resp. a strong solution of the initial value problem):

(10) $$V(0)=V(T) \qquad \text{in } H_\sigma(B)\times L^2(B),$$

(11) $$V(0)={}^t(\tilde{a},\tilde{h}) \qquad \text{in } H_\sigma(B)\times L^2(B),$$

where a and h are certain prescribed initial data in $H_\sigma^1(\Omega(0))\times\mathring{W}_2^1(\Omega(0))$.

§4. Proof of the theorem. We prove Theorem B. Let us put $U={}^t(u,\theta)={}^t(v-w,\Theta-\psi)=V-W$. Then the assertions (i) and (ii) of Theorem B can be reduced to the global existence of the strong solution and the decay problem of it for the next initial value problem in $\hat{\Omega}=\bigcup_{0<t<\infty}\Omega(t)\times\{t\}$:

(12) $$\begin{cases} u_t+(u\cdot\nabla)u=-\nabla p-(u\cdot\nabla)w-(w\cdot\nabla)u-(u\cdot\nabla)b-(b\cdot\nabla)u-R\theta+\Delta u \\ \quad \operatorname{div} u=0 \\ \theta_t+(u\cdot\nabla)\theta=(1/P)\Delta\theta-(u\cdot\nabla)\psi-(w\cdot\nabla)\theta-(u\cdot\nabla)\bar{\theta}-(b\cdot\nabla)\theta \end{cases}$$

(13) $$u|_{\partial\Omega(t)}=0,\quad \theta|_{\partial\Omega(t)}=0 \qquad \text{for any } t\in(0,\infty),$$

(14) $$u|_{t=0}=u_0,\quad \theta|_{t=0}=\theta_0 \qquad \text{in } \Omega(0).$$

Here let put

(15) $$\|\tilde{f}\|_{2,\infty,T}^2=\sup_{0\leq t\leq T-1}\int_t^{t+1}\|\tilde{f}(s)\|_B^2\,ds.$$

Then we note the following estimate given in [4]:

(16) $$\varphi^t(W(t))\leqq\gamma_0,$$

where γ_0 is a constant depending on $\|\tilde{f}\|_{2,\infty,T}$. Moreover, it holds also that $\gamma_0 \to 0$ as $\|\tilde{f}\|_{2,\infty,T} \to 0$. Employing (16), the claim (i) can be proven by using the similar arguments to those in [4]. So we omit details.

To show (ii), multiplying both sides of (12) by U and integrating over $\Omega(t)$, we get by standard inequalities

$$(17) \qquad \frac{1}{2}\frac{d}{dt}\|U(t)\|^2 + 2\varphi^t(U(t)) \leqq C\|\nabla U(t)\|^2 \cdot (\|\nabla W(t)\| + \|\nabla b(t)\| + \|\nabla \bar{\theta}(t)\| + |R|).$$

Considering (16), if b and $\bar{\theta}$ are sufficiently small and ν is sufficiently large, there exists $\delta > 0$ such that $2-\delta > 0$ and

$$(18) \qquad \frac{1}{2}\frac{d}{dt}\|U(t)\|^2 + (2-\delta)\varphi^t(U(t)) \leqq 0$$

hold. Hence we have proven (ii). Q.E.D.

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