20. *Minimal Currents and Relaxation of Variational Integrals on Mappings of Bounded Variation*

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1990)

1. Introduction and main results. Let T be a 1-dimensional current of locally finite mass on $\boldsymbol{R}^m$. By the Riesz representation theorem T is identified with a $\boldsymbol{R}^m$-valued Radon measure $T=(T^1, \cdots, T^m)$ on $\boldsymbol{R}^m$ (see e.g., [5,12]). If $F=F(y, \eta)$ is a nonnegative continuous function on $\boldsymbol{R}^m \times \boldsymbol{R}^m$ and is positively homogeneous of degree one in η, a new measure $F(y, T)$ is associated with T (cf. [10]). We consider a functional

$$I_F(T)=\int_{\boldsymbol{R}^m} F(y, T). \tag{1}$$

Here F is assumed to be convex in η and satisfy a growth condition

$$k|\eta| \leq F(y, \eta) \leq K|\eta| \tag{2}$$

with $K \geq k > 0$ independent of y and η. If T is a current representing an oriented C^1 curve C, $I_F(T)$ is the length of the curve C with metric density F, so $I_F(T)$ agrees with the standard length of C in $\boldsymbol{R}^m$ when $F(y, \eta)=|\eta|$.

We call S a *minimal current* from $a \in \boldsymbol{R}^m$ to $b \in \boldsymbol{R}^m$ if

$$I_F(S)=\inf\{I_F(T)\,;\, T \in \mathcal{M}_1 \text{ and } \partial T=\delta_b-\delta_a\}.$$

Here δ_a denotes the Dirac measure supported at a and ∂T denotes the boundary of T, *i.e.* $\partial T=\operatorname{div} T$. The space $\mathcal{M}_1$ represents the set of all 1-currents of locally finite mass in $\boldsymbol{R}^m$. Our main result on minimal currents asserts that a shortest curve is a minimal current.

Theorem 1. *There exists a current representing a simple Lipschitz curve from a to b which is a minimal current. In particular,*

$$\inf_{\substack{\partial T=\delta_b-\delta_a \\ T \in \mathcal{M}_1}} I_F(T)=\inf\left\{\int_0^1 F(\gamma(t), \dot{\gamma}(t))dt\,;\, \gamma:[0,1] \longrightarrow \boldsymbol{R}^m \text{ is Lipschitz and } \gamma(0)=a,\ \gamma(1)=b\right\} \quad (\dot{\gamma}=d\gamma/dt). \tag{3}$$

If $F(y, \eta)$ is independent of y, we have proved in [2, Lemma 8.3] that the straight line from a to b is a minimal current. Theorem 1 has important applications in relaxations of variational integrals on $BV(\Omega, \boldsymbol{R}^m)$, the set of mapping $u: \Omega \rightarrow \boldsymbol{R}^m$ of bounded variation, where Ω is an open set in $\boldsymbol{R}^n$.

We consider a functional $\mathcal{F}$ of C^1 mapping $u: \Omega \rightarrow \boldsymbol{R}^m$

$$\mathcal{F}(u)=\int_\Omega f(x, u(x), \nabla u(x))dx.$$

The density function $f=f(x, y, \xi)$ we discuss here is a nonnegative continuous function in $\Omega \times \boldsymbol{R}^m \times \boldsymbol{R}^{nm}$ and convex in ξ. Here the Jacobi matrix $\nabla u(x)$ of u at x is identified with an element of $\boldsymbol{R}^{nm}$. We do not assume homoge-

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nuity but a growth condition

$$k|\xi|\leq f(x,y,\xi)\leq K(|\xi|+1).$$

Under these conditions it is well-known that the recession function

$$f_{\infty}(x,y,\xi)=\underline{\lim}_{t\downarrow 0} f(x,y,\xi/t)t$$

exists and has the homogenuity in ξ as well as all other properties of f. For technical reasons we further assume the following equicontinuity. For every $(x_0, y_0)\in\Omega\times\boldsymbol{R}^m$ and $\varepsilon>0$ there is $\delta>0$ such that $|x-x_0|$, $|y-y_0|<\delta$ implies

$$|f(x,y,\xi)-f(x_0,y_0,\xi)|\leq\varepsilon(1+|\xi|).$$

Let $\overline{\mathcal{F}}$ be the lower semicontinuous L^1_{loc} *relaxation****) of $\mathcal{F}$ on $BV(\Omega,\boldsymbol{R}^m)$, that is

$$\overline{\mathcal{F}}(u)=\inf\{\underline{\lim}_{l\to\infty}\mathcal{F}(u_l);\ u_l\longrightarrow u \text{ in } L^1_{\text{loc}}(\Omega,\boldsymbol{R}^m) \text{ and } u_l \text{ is } C^1\}.$$

Our problem is to find an explicit representation of $\overline{\mathcal{F}}$ for $u\in BV(\Omega,\boldsymbol{R}^m)$. This question is posed by De Giorgi [4]. When f does not depend on y this problem is solved by [6, 8, 10]. If f depends on y, so far only the cases $m=1$ and $n=1$ were settled by [3] and [11], respectively.

We shall answer to this problem for arbitrary n, $m\geq 1$ assuming that f satisfies an isotropy condition

$$f(x,y,(\xi_i^j))\geq f\left(x,y,\left(\sum_{h=1}^{n} q_h\xi_h^j q_i\right)\right), \tag{4}$$

where $q=(q_1,\cdots,q_n)\in\boldsymbol{R}^n$ and $\xi=(\xi_i^j)\in\boldsymbol{R}^{nm}$, $1\leq i\leq n$, $1\leq j\leq m$. For $u\in BV(\Omega,\boldsymbol{R}^m)$ it is well-known [5, 7, 12] that ∇u is a (matrix) Radom measure decomposed as

$$\nabla u=\nabla u\lfloor\Omega_0+\nabla u\lfloor(\Omega-\Omega_0-\Sigma)+\nu\otimes(u^+-u^-)\mathcal{H}^{n-1}\lfloor\Sigma.$$

Here Σ denotes the set of jump discontinuities of u and ν represents a unit normal to Σ. The functions $u^{\pm}$ are the trace of u on Σ defined by $u^{\pm}(x)=\underline{\lim}_{\varepsilon\downarrow 0} u(x\pm\varepsilon\nu(x))$ and $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. By $\mu\lfloor A$ we mean a measure on Ω defined by $(\mu\lfloor A)(B)=\mu(A\cap B)$ for $B\subset\Omega$, where μ is a measure. For a, $b\in\boldsymbol{R}^m$ and $q\in\boldsymbol{R}^n$ we introduce a distance like function:

$$\begin{aligned} D_x(a,b,q)=\inf\Big\{&\int_0^1 f_{\infty}(x,\gamma(t),q\otimes\dot{\gamma}(t))dt;\\ &\gamma:[0,1]\longrightarrow\boldsymbol{R}^m \text{ is Lipschitz and } \gamma(0)=a,\gamma(1)=b\Big\}\end{aligned} \tag{5}$$

A combination of Theorem 1 and results in [2] yield our main result for relaxation of $\mathcal{F}$ when f sastisfies all above assumptions. By $|\mu|$ we mean the total variation measure of μ and $d\mu/d|\mu|$ denotes the Radon-Nikodym derivative.

Theorem 2. *For* $u\in BV(\Omega,\boldsymbol{R}^m)$ *it holds*

$$\begin{aligned}\overline{\mathcal{F}}(u)=&\int_{\Omega_0} f(x,u(x),\nabla u(x))dx+\int_{\Omega-\Omega_0-\Sigma} f_{\infty}\left(x,u(x),\frac{d\nabla u}{d|\nabla u|}(x)\right)|\nabla u|\\ &+\int_{\Sigma} D_x(u^-(x),u^+(x),\nu(x))d\mathcal{H}^{n-1}(x).\end{aligned} \tag{6}$$

***) This terminology is due to De Giorgi [4]. It is also called the lower semicontinuous envelope.

In this note we just give a brief sketch of proofs; the details will be published elsewhere.

After this work is completed, we are informed of a recent work of Ambrosio, Mortola and Tortorelli [1] which proves only "$\geq$" in (6) of Theorem 2 without (4). Moreover, they show that equality in (6) does not necessarily hold without assuming (4).

2. Discritization and networks. We approximate a current connecting a and b by *real polyhedral chain* (see [5] for the definition).

Lemma 3. *Suppose that $T \in \mathcal{M}_1$ satisfies $\partial T=\delta_b-\delta_a$, $a, b \in \boldsymbol{R}^m$ and that its total mass $M(T)$ is finite. There is a sequence of real polyhedral chain $T_\varepsilon \in \mathcal{M}_1$ with $\partial T_\varepsilon=\delta_b-\delta_a$ such that T_ε converges weakly to T and that $M(T_\varepsilon) \to M(T)$ as $\varepsilon\to 0$.*

Sketch of the proof. We take $L \in \mathcal{M}_1$ representing a piecewise linear curve from a to b such that $M(T)=M(L)+M(R)$ with $R=T-L$. We may assume that R is smooth by a standard mollification. Since $\partial R=0$, Poincaré's lemma implies that there is a smooth 2-current Φ such that $R=\partial\Phi$. We next approximate Φ by a piecewise linear Ψ with compact support associated with a simplicial decomposition of a large cube. For simplicity we only discuss the case $m=2$ so that Ψ is a scalar function. We approximate Ψ by a piecewise constant function

$$\Psi_\varepsilon(x)=k\varepsilon \quad \text{if } \theta+k\varepsilon\leq\Psi(x)<(k+1)\varepsilon+\theta, \quad k\text{: integer}$$

so that $\Psi_\varepsilon\to\Psi$ and $M(\partial\Psi_\varepsilon)\to M(\partial\Psi)$ as $\varepsilon\to 0$. We take $\theta \in \boldsymbol{R}$ such that

$$M(L+\partial\Psi_\varepsilon)=M(L)+M(\partial\Psi_\varepsilon).$$

We thus find a desired approximation $T_\varepsilon=L+\partial\Psi_\varepsilon$.

Sketch of the proof of Theorem 1. Let $\{T_j\}$ be a minimizing sequence of

$$\inf\{I_F(T)\,;\ T \in \mathcal{M}_1,\ \partial T=\delta_b-\delta_a\}. \tag{7}$$

By Lemma 3 we approximate T_j by a real polyhedral chain $T_{j\varepsilon}$. Let P denote the support of $T_{j\varepsilon}$. Since P is regarded as a network, applying the theory of minimal flow problem (see e.g. [9]) to

$$\inf\{I_F(T)\,;\ T \text{ is real polyhedral chain supported in } P \text{ and } \partial T=\delta_b-\delta_a\}$$

we see the infimum is attained at multiplicity one current $S_{j\varepsilon}$ representing a Lipschitz curve from a to b. By Reshetnyak's continuity theorem [10] $M(T_{j\varepsilon})\to M(T_j)$ with (2) implies $I_F(T_{j\varepsilon})\to I_F(T_j)$ as $\varepsilon\to 0$. We now observe that $S_{j\varepsilon}$ is a minimizing sequence of (7) by taking a subsequence $\varepsilon=\varepsilon_j\to 0$ since $I_F(S_{j\varepsilon})\leq I_F(T_{j\varepsilon})$. This proves (3). By a standard compactness argument and (2) we see the infimum of the right hand side of (3) is attained at a simple Lipschitz curve from a to b.

3. Skecth of the proof of Theorem 2. We shall prove "$\geq$" in (6). Let $\tilde{D}_x$ denote

$$\tilde{D}_x(a, b, \nu)=\inf\left\{\int_{\boldsymbol{R}^m} f_\infty(x, y, (S_i^j))\,;\ \partial S_i=\nu_i(\delta_b-\delta_a),\ S_i \in \mathcal{M}_1,\ 1\leq i\leq n\right\}.$$

From main results in [2, Theorems 5.1 and 8.1] it follows that

$$\overline{\mathcal{F}}(u)=\int_{\Omega_0} f(x, u, \nabla u)dx+\int_{\Omega-\Omega_0-\Sigma} f_\infty\left(x, u, \frac{d\nabla u}{d|\nabla u|}\right)|\nabla u|+\int_\Sigma \theta(x)d\mathcal{H}^{n-1}(x)$$

with some θ satisfying $\theta(x) \geq \tilde{D}_x(u^-(x), u^+(x), \nu(x))$.

Applying (4) and Theorem 1 with $F(y, \eta) = f_\infty(x, y, \nu(x) \otimes \eta)$ in (1) yields $\tilde{D}_x(a, b, \nu) \geq D_x(a, b, \nu)$, where D_x is defined in (5). We thus prove "$\geq$" in (6).

The converse inequality is proved by approximating u by piecewise constant functions. We note that this part is independently proved by [1].

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