45. Some Trace Relations of Twisting Operators on the Spaces of Cusp Forms of Half-integral Weight

By Masaru UEDA
Department of Mathematics, Kyoto University

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In the papers [3] and [4], we calculated the traces of Hecke operators $\tilde{T}(n^2)$ on the space of cusp forms of half-integral weight $S(k+1/2, N, \chi)$ and on the Kohnen subspace $S(k+1/2, N, \chi)_K$. Moreover we found that the above traces are linear combinations of the traces of certain operators on the spaces $S(2k, N')$ (N' runs over divisors of $N/2$). In this paper, we report similar trace relations of the twisting operators on the spaces $S(k+1/2, N, \chi)$ and $S(k+1/2, N, \chi)_K$. Details will appear in [5].

Preliminaries. (a) **General notations.** Let k denote a positive integer. If $z \in \boldsymbol{C}$ and $x \in \boldsymbol{C}$, we put $z^x = \exp(x \cdot \log(z))$ with $\log(z) = \log(|z|) + \sqrt{-1}\arg(z)$, $\arg(z)$ being determined by $-\pi < \arg(z) \leq \pi$. Also we put $\boldsymbol{e}(z) = \exp(2\pi\sqrt{-1}z)$.

Let $\mathfrak{H}$ be the complex upper half plane. For a complex-valued function $f(z)$ on $\mathfrak{H}$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\boldsymbol{R})$, $\gamma = \begin{pmatrix} u & v \\ w & x \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathfrak{H}$, we define functions $J(\alpha, z)$, $j(\gamma, z)$ and $f|[\alpha]_k(z)$ on $\mathfrak{H}$ by: $J(\alpha, z) = cz + d$, $j(\gamma, z) = \left(\frac{-1}{x}\right)^{-1/2} \left(\frac{w}{x}\right)(wz+x)^{1/2}$ and $f|[\alpha]_k(z) = (\det \alpha)^{k/2} J(\alpha, z)^{-k} f(\alpha z)$.

For a real number x, $[x]$ means the greatest integer m with $x \geq m$. $|\ |_p$ is the p-adic absolute value which is normalized with $|p|_p = p^{-1}$. See [1, p. 82] for the definition of the Kronecker symbol $\left(\frac{a}{b}\right)$ (a, b integers with $(a, b) \neq (0, 0)$). Let N be a positive integer and m an integer $\neq 0$. We write $m | N^\infty$ if every prime factor of m divides N. For a finite-dimensional vector space V over $\boldsymbol{C}$ and a linear operator T on V, $\mathrm{tr}(T|V)$ denotes the trace of T on V.

(b) **Modular forms of integral weight.** Let N be a positive integer. By $S(2k, N)$, we denote the space of all holomorphic cusp forms of weight $2k$ with the trivial character on the group $\Gamma = \Gamma_0(N)$.

Let $\alpha \in GL_2^+(\boldsymbol{R})$. If Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, we define a linear operator $[\Gamma\alpha\Gamma]_{2k}$ on $S(2k, N)$ by: $f|[\Gamma\alpha\Gamma]_{2k} = (\det \alpha)^{k-1} \sum_{\alpha_i} f|[\alpha_i]_{2k}$, where α_i runs over a system of representatives for $\Gamma \backslash \Gamma\alpha\Gamma$. For a natural number n with $(n, N) = 1$, we put $T(n) = T_{2k,N}(n) = \sum_{ad=n} \left[\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma\right]_{2k}$, where the sum is extended over all pairs of integers (a, d) such that $a, d > 0$, $a | d$, $ad = n$. Moreover let Q be a positive divisor of N such that $(Q, N/Q) = 1$ and $Q \neq 1$.

Take an element $\gamma(Q)\in SL_2(\boldsymbol{Z})$ which satisfies the conditions:

$$\gamma(Q)\equiv\begin{cases}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} & (\operatorname{mod} Q);\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & (\operatorname{mod} N/Q).\end{cases}$$

Put $W(Q)=\gamma(Q)\begin{pmatrix}Q & 0\\ 0 & 1\end{pmatrix}$. Then $W(Q)$ is a normalizer of Γ and $[W(Q)]_{2k}$ induces a linear operator on $S(2k, N)$.

(c) **Modular forms of half-integral weight.** Let N be a positive integer divisible by 4 and χ an even character modulo N such that $\chi^2=1$. Put $\mu=\operatorname{ord}_2(N)$, $M=2^{-\mu}N$ and $\Gamma=\Gamma_0(N)$.

Let $\mathfrak{G}(k+1/2)$ be the group consisting of pairs (α, φ), where $\alpha=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2^+(\boldsymbol{R})$ and φ is a holomorphic function on $\mathfrak{H}$ satisfying $\varphi(z)=t(\det\alpha)^{-k/2-1/4}J(\alpha, z)^{k+1/2}$ with $t\in\boldsymbol{C}$ and $|t|=1$. The group law is defined by: $(\alpha, \varphi(z))\cdot(\beta, \psi(z))=(\alpha\beta, \varphi(\beta z)\psi(z))$. For a complex-valued function f on $\mathfrak{H}$ and $(\alpha, \varphi)\in\mathfrak{G}(k+1/2)$, we define a function $f|(\alpha, \varphi)$ on $\mathfrak{H}$ by: $f|(\alpha, \varphi)(z)=\varphi(z)^{-1}f(\alpha z)$.

By $\Delta=\Delta_0(N, \chi)_{k+1/2}$, we denote the subgroup of $\mathfrak{G}(k+1/2)$ consisting of all pairs (γ, φ), where $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\gamma\in\Gamma$ and $\varphi(z)=\chi(d)j(\gamma, z)^{2k+1}$. We denote by $S(k+1/2, N, \chi)$ the space of all complex-valued holomorphic functions f on $\mathfrak{H}$ which satisfy $f|\xi=f$ for all $\xi\in\Delta$ and which are holomorphic and vanish at all cusps of Γ. When $\mu=2$, we define the Kohnen subspace $S(k+1/2, N, \chi)_K$ as follows:

$$S\left(k+\frac{1}{2}, N, \chi\right)_K=\left\{\begin{matrix}S\left(k+\frac{1}{2}, N, \chi\right)\ni f(z)=\sum_{n=1}^{\infty}a(n)\boldsymbol{e}(nz);\\ a(n)=0 \text{ for } \varepsilon(-1)^k n\equiv 2, 3 \pmod 4\end{matrix}\right\}.$$

Here, $\varepsilon=\chi_2(-1)$ where χ_2 is the 2-primary component of χ.

Let $\xi\in\mathfrak{G}(k+1/2)$. If Δ and $\xi^{-1}\Delta\xi$ are commensurable, we define a linear operator $[\Delta\xi\Delta]_{k+1/2}$ on $S(k+1/2, N, \chi)$ by: $f|[\Delta\xi\Delta]_{k+1/2}=\sum_\eta f|\eta$, where η runs over a system of representatives for $\Delta\backslash\Delta\xi\Delta$.

Then for a natural number n with $(n, N)=1$, we put

$$\tilde{T}(n^2)=\tilde{T}_{k+1/2,N,\chi}(n^2)=n^{k-3/2}\sum_{ad=n}a\left[\Delta\left(\begin{pmatrix}a^2 & 0\\ 0 & d^2\end{pmatrix}, (d/a)^{k+1/2}\right)\Delta\right]_{k+1/2},$$

where the sum is extended over all pairs of integers (a, d) such that $a, d>0$, $a|d$ and $ad=n$. Then $S(k+1/2, N, \chi)_K$ is invariant under the action of the operators $\tilde{T}(n^2)$. Hence, we can consider the traces of those operators on $S(k+1/2, N, \chi)_K$.

From now on until the end of this paper, we assume the following:

Assumption. *ψ is a non-trivial primitive character such that $\psi^2=1$ and the conductor of ψ, say L, is odd and $L^2|N$.*

We fix the notations L and ψ in the above assumption. Furthermore, we decompose N as follows: $N=L_0L_1$, $L_1=2^{\operatorname{ord}_2(N)}L_2$, where $L_0>0$, $L_1>0$, $L_0|L^\infty$, and $(L_1, L)=1$. From this assumption and the fact $\chi^2=1$, it follows that the conductor of χ divides (N/L). From [2, Lemma 3.6], we can consider the linear operator R_ψ on $S(k+1/2, N, \chi)$: $f(z)=\sum_{n=1}^{\infty} a(n)e(nz) \mapsto f|R_\psi(z) := \sum_{n=1}^{\infty} \psi(n)a(n)e(nz)$. We call R_ψ the twisting operator for ψ.

Statement of results. We use the above notations and also for a prime divisor p of N, $\mathrm{ord}_p(N)=\nu_p=\nu$ or μ, according as p is odd or $p=2$. Put $M=2^{-\mu}N$ and $N_0=\prod_{q|L} q^{2[(\nu-1)/2]+1}$. Moreover we use the following notations:

For any odd prime number p and any integers a, b $(0\leq a\leq \nu/2)$, we put

$$\lambda(p, b; a)=\begin{cases} 1, & \text{if } a=0; \\ 1+\left(\dfrac{-b}{p}\right), & \text{if } 1\leq a\leq [(\nu-1)/2]; \\ \chi_p(-b), & \text{if } \nu \text{ is even and } a=\nu/2, \end{cases}$$

where χ_p is the p-primary component of χ. For any integer b and any square divisor c of M, we put

$$\Lambda(b; c) := \prod_{p|M} \lambda(p, b; \mathrm{ord}_p(c)/2).$$

Theorem. *Let N be a positive integer such that $2\leq\mu=\mathrm{ord}_2(N)\leq 4$, and χ an even character modulo N such that $\chi^2=1$ and the conductor of χ is divisible by 8 if $\mu=4$. Let n be any positive integer such that $(n, N)=1$. Then we have the following trace relations* (1)–(2).

(1) *Suppose that $k\geq 2$. Then we have:*

$$\mathrm{tr}\,(R_\psi \tilde{T}(n^2)\,|\,S(k+1/2, N, \chi))$$
$$=\left(\frac{-1}{L}\right)^k \chi_{L_0}(n)\chi_{L_1}(-L) \sum_{N_1} \Lambda(Ln; N_1)\,\mathrm{tr}\,([W(N_0N_1)]T(n)\,|\,S(2k, 2^{\mu-1}N_0N_1N_2)).$$

(2) *Suppose that $k\geq 2$ and $N=4M$. Then we have:*

$$\mathrm{tr}\,(R_\psi \tilde{T}(n^2)\,|\,S(k+1/2, N, \chi)_K)$$
$$=\left(\frac{-1}{L}\right)^k \chi_{L_0}(n)\chi_{L_1}(-L) \sum_{N_1} \Lambda(Ln; N_1)\,\mathrm{tr}\,([W(N_0N_1)]T(n)\,|\,S(2k, N_0N_1N_2)).$$

Here, N_1 in the sum $\sum_{N_1}$ runs over all square divisors of L_2 and $N_2=L_2\prod_{p|N_1}|L_2|_p$. χ_{L_0} (resp. χ_{L_1}) is the L_0 (resp. L_1)-primary component of χ.

Remark. We also have some similar relations for the case of $k=1$, or $\mu\geq 5$, or etc. (cf. [5] § 4).

Supplementary remarks. In the case of the twisting operator, we have the same phenomena as in the case of the Hecke operators (cf. [3], [4]).

(1) When the 2-order of N $(=\mathrm{ord}_2(N)=\mu)$ is small (for example $\mu\leq 3$), cusp forms of half-integral weight $k+1/2$ of level N correspond to those of integral weight $2k$ *of level* $N/2$.

(2) On the other hand, when μ is big (for example $\mu\geq 8$), cusp forms of weight $k+1/2$ of level N correspond to those of weight $2k$ *of level at most* $N/4$.

We do not know why this difference occurs.

References

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