# 75. On Fractional Powers of a Class of Elliptic Differential Operators with Feedback Boundary Conditions

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§1. **Introduction.** Control theoretic studies of elliptic differential operators constitute a rapidly growing area of partial differential equations. The area has a substantial possibility of producing many interesting problems of p.d. equations. In these studies, bounded or unbounded feedback operators enter the given elliptic operator in any form. We consider in this paper a typical elliptic system $(\mathcal{L}, \tau)$ in a connected bounded domain $\Omega$ of $\mathbb{R}^m$ with a finite number of smooth boundaries $\Gamma$ of $(m-1)$-dimension. More precisely, let $\mathcal{L}$ denote a uniformly elliptic differential operator of order 2 in $\Omega$ described by

$$\mathcal{L}u=-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{m}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u,$$

where $a_{ij}(x)=a_{ji}(x)$, $1\le i, j\le m$, and $x\in\bar{\Omega}$. Associated with $\mathcal{L}$ is a generalized Neumann boundary operator $\tau$ described by

$$\tau u=\frac{\partial u}{\partial \nu}+\sigma(\xi)u,$$

where $\partial/\partial\nu=\sum_{i,j=1}^{m}a_{ij}(\xi)\nu_i(\xi)\partial/\partial x_j$, and $(\nu_1(\xi), \cdots, \nu_m(\xi))$ indicates the outward normal at $\xi\in\Gamma$. Let us define a linear operator $L$ in $L^2(\Omega)$ by

$$Lu=\mathcal{L}u,\quad u\in\mathcal{D}(L)=\{u\in H^2(\Omega)\,;\ \tau u=0 \text{ on } \Gamma\}.$$

All norms hereafter will be $L^2(\Omega)$- or $\mathcal{L}(L^2(\Omega))$-norms unless otherwise indicated. Necessary regularity of the coefficient functions in $L$ is of course assumed. As is well known [3], there is a sector $\bar{\Sigma}_{-\alpha}=\bar{\Sigma}-\alpha$, $\alpha>0$, such that $\bar{\Sigma}_{-\alpha}$ is contained in $\rho(L)$, and

$$\|(\lambda-L)^{-1}\|\le\frac{\text{const}}{1+|\lambda|},\quad \lambda\in\bar{\Sigma}_{-\alpha},$$

where $\bar{\Sigma}=\{\lambda\,;\ \theta\le|\arg\lambda|\le\pi\}$, $0<\theta<\pi/2$, and the upper bar indicates the closure of a set. Choose a positive constant $c(>\alpha)$, and set $L_c=L+c$. Then fractional powers of the operator $L_c$ are well defined. As is well known [2], we have a relation

$$\mathcal{D}(L_c^{\omega})=H^{2\omega}(\Omega),\quad 0\le\omega<3/4 \tag{1}$$

with equivalent norms. The relation has played an important role in the study of boundary control systems.

Let us introduce an operator $M$ as

$$Mu=\mathcal{L}u,\quad u\in\mathcal{D}(M)=\left\{u\in H^2(\Omega)\,;\ \tau u=\sum_{k=1}^{p}\langle u, w_k\rangle h_k \text{ on } \Gamma\right\}. \tag{2}$$

Here, $\langle\cdot,\cdot\rangle$ indicates the inner product in $L^2(\Omega)$, $w_k$ given observation weighting functions belonging to $L^2(\Omega)$, and $h_k$ actuators belonging to $H^{1/2}(\Gamma)$. Thus the boundary condition is described in feedback form. The operator $M$ is connected with the stabilization problem of the differential equation in $L^2(\Omega)$;

$$(3) \qquad \frac{du}{dt}+Mu=0, \quad u(0)=u_0.$$

It is the purpose of the paper to derive a generalization of the relation (1) to fractional powers of $M$. The generalization is fundamental in analysis and synthesis of eqn. (3) Some preliminary results are stated as follows:

**Proposition 1.1.** *The operator $M$ is closed, and the adjoint operator of $M$ is given by the formula*

$$(4) \qquad \begin{gathered} M^*v=\mathcal{L}^*v-\sum_{k=1}^{p}\langle v, h_k\rangle_\Gamma w_k, \\ v\in\mathcal{D}(M^*)=\mathcal{D}(L^*)=\{v\in H^2(\Omega)\,;\ \tau^*v=0 \text{ on } \Gamma\}, \end{gathered}$$

*where the pair $(\mathcal{L}^*,\tau^*)$ indicates the formal adjoint of $(\mathcal{L},\tau)$, and $\langle\cdot,\cdot\rangle_\Gamma$ the inner product in $L^2(\Gamma)$.*

It is standard to show that there is a $\beta(>\alpha)$ such that $\bar{\Sigma}_{-\beta}=\bar{\Sigma}-\beta$ is contained in $\rho(M^*)$ (and therefore in $\rho(M)$), and that

$$\|(\lambda-M)^{-1}\|=\|(\bar{\lambda}-M^*)^{-1}\|\le\frac{\text{const}}{1+|\lambda|}, \quad \lambda\in\bar{\Sigma}_{-\beta}.$$

Thus $-M$ generates an analytic semigroup $\exp(-tM)$, $t>0$. Given a $\mu>0$, it is possible to choose an integer $p$, $w_k$, and $h_k$, $1\le k\le p$ so that the closed region $\bar{\Sigma}_{-\beta}\cup\{\lambda;\,\mathrm{Re}\,\lambda\le\mu\}$ is contained in $\rho(M)$, e.g., [7]. This is a stabilization result for eqn. (3), and is our assumption throughout the paper (there are many other works on stabilization in existing literature). Fractional powers of the operator $M$ are well defined.

## § 2. Main result.

Our main result is simply stated as follows:

**Theorem 2.1.** *The relation $\mathcal{D}(M^\omega)=\mathcal{D}(L_c^\omega)=H^{2\omega}(\Omega)$, $0\le\omega<1/2$ holds algebraically and topologically. In addition, if $w_k\in H^{1/2}(\Omega)$, $1\le k\le p$, then the equivalence relation holds for $\omega$, $0\le\omega<3/4$.*

**Remark.** A similar but more restrictive result holds for $M$ with the boundary condition replaced by $\tau u=\sum_{k=1}^{p}\langle u, w_k\rangle_\Gamma h_k$.

*Outline of the proof.* Let us consider the case where $w_k\in H^{1/2}(\Omega)=\mathcal{D}(L_c^{1/4})$. The other case is similarly treated. The original proof [2] of the relation (1) is based on examining local regularity of a class of functions in $\Omega$ near the boundary $\Gamma$. It seems, however, difficult to prove our theorem along the same line. For a given $g\in H^{1/2}(\Gamma)$, the boundary value problem described by

$$\mathcal{L}_c u=0 \quad \text{in } \Omega, \quad \text{and} \quad \tau u=g \quad \text{on } \Gamma$$

admits a unique solution $u\in H^2(\Omega)$, which is denoted by $Ng$. It is easy to see that the operator $N$ belongs to $\mathcal{L}(H^{1/2}(\Gamma)\,;H^2(\Omega))$. Let $u(t)=\exp(-tM)u_0$ be an arbitrary solution to eqn. (3). For any $\vartheta$, $1/2<\vartheta<3/4$, set $v(t)=L_c^{-\vartheta}u(t)$. Then we see that $v(t)$ satisfies a differential equation in $L^2(\Omega)$

$$\frac{dv}{dt}+(L-F)v=0, \quad v(0)=v_0=L_c^{-\vartheta}u_0,$$

where $Fv=\sum_{k=1}^{p}\langle L_c^{\vartheta}v, w_k\rangle L_c^{1-\vartheta}Nh_k$. The operator $F$ is subordinate to $L_c^{\vartheta-1/4}$ since $w_k\in\mathcal{D}(L_c^{*1/4})=H^{1/2}(\Omega)$.

**Lemma 2.2.** *The operator $L-F$ has a compact resolvent. There is a $\gamma>0$ such that $\bar{\Sigma}_{-\gamma}\cup\{\lambda;\operatorname{Re}\lambda<\mu\}$ is contained in $\rho(L-F)$, and that*

$$\|(\lambda-L+F)^{-1}\|\le\frac{\text{const}}{1+|\lambda|}, \quad \lambda\in\bar{\Sigma}_{-\gamma}\cup\{\lambda;\operatorname{Re}\le\mu_1\}.$$

*for each $\mu_1$, $0<\mu_1<\mu$.*

It is not difficult to show that

$$(5) \qquad (\lambda-M)^{-1}=L_c^{\vartheta}(\lambda-L+F)^{-1}L_c^{-\vartheta}$$

for $\operatorname{Re}\lambda<\mu$. The right-hand side of eqn. (5) is analytic in $\lambda\in\rho(L-F)$. Thus, $(\lambda-M)^{-1}$ has an extension to an operator analytic in $\lambda\in\rho(L-F)$. The extension is, however, nothing but the resolvent of $M$ [1]. We have therefore shown that $\rho(L-F)$ is contained in $\rho(M)$, and that eqn. (5) holds for $\lambda\in\rho(L-F)$.

Let us consider fractional powers of $M$ and $L-F$. According to (5) valid for $\lambda\in\rho(L-F)$, we can show that

$$(6) \qquad M^{-\vartheta}=L_c^{\vartheta}(L-F)^{-\vartheta}L_c^{-\vartheta}.$$

**Lemma 2.3.** *The equivalence relation $\mathcal{D}((L-F)^{\omega})=\mathcal{D}(L_c^{\omega})$, $0\le\omega<3/4+\vartheta$ holds algebraically and topologically.*

According to Lemma 2. 3, we see that

$$L_c^{\vartheta}(L-F)^{\vartheta}L_c^{-2\vartheta}=L_c^{\vartheta}(L-F)^{-\vartheta}(L-F)^{2\vartheta}L_c^{-2\vartheta}\in\mathcal{L}(L^2(\Omega)),$$

since $2\vartheta<3/4+\vartheta$. Thus, $L_c^{\vartheta}(L-F)^{\vartheta}L_c^{-\vartheta}u$ is well defined as an element of $L^2(\Omega)$ for any $u\in\mathcal{D}(L_c^{\vartheta})$, and by the relation (6)

$$M^{-\vartheta}(L_c^{\vartheta}(L-F)^{\vartheta}L_c^{-\vartheta}u)=u, \quad \text{or} \quad M^{\vartheta}u=L_c^{\vartheta}(L-F)^{\vartheta}L_c^{-\vartheta}u,$$

which shows that $\mathcal{D}(L_c^{\vartheta})$ is contained in $\mathcal{D}(M^{\vartheta})$, and that

$$\|M^{\vartheta}u\|\le\text{const}\,\|L_c^{\vartheta}u\|, \quad u\in\mathcal{D}(L_c^{\vartheta}).$$

As to the converse relation, set $v=M^{\vartheta}u$ for $u\in\mathcal{D}(M^{\vartheta})$. Then,

$$\begin{aligned} u&=L_c^{\vartheta}(L-F)^{-\vartheta}L_c^{-\vartheta}v \qquad \text{by (6)} \\ &=L_c^{-\vartheta}L_c^{2\vartheta}(L-F)^{-2\vartheta}(L-F)^{\vartheta}L_c^{-\vartheta}v\in\mathcal{D}(L_c^{\vartheta}), \end{aligned}$$

which proves that $\mathcal{D}(M^{\vartheta})$ is contained in $\mathcal{D}(L_c^{\vartheta})$, and that

$$\|L_c^{\vartheta}u\|\le\text{const}\,\|M^{\vartheta}u\|, \quad u\in\mathcal{D}(M^{\vartheta}).$$

Therefore, we have shown that $\mathcal{D}(M^{\vartheta})=\mathcal{D}(L_c^{\vartheta})$ with equivalent norms for any $\vartheta$, $1/2<\vartheta<3/4$.

The proof of the relation $\mathcal{D}(M^{\omega})=\mathcal{D}(L_c^{\omega})$ for $0\le\omega\le1/2$ is carried out as follows: It $k>0$ is chosen large enough, the operator $M_k=M+k$ is $m$-accretive. Thus we have

**Lemma 2.4.** *The operator $M_k^{\omega}$ is $m$-accretive for $0<\omega<1$.*

It is similarly shown that the operator $L_c^{\omega}$, $0<\omega<1$ is $m$-accretive. We remark that $\mathcal{D}(M_k^{\omega})=\mathcal{D}(M^{\omega})$, $0\le\omega\le1$ with equivalent norms. For a fixed $\vartheta$, $1/2<\vartheta<3/4$, a generalization of the Heinz inequality [4] is applied to $M_k^{\vartheta}$ and $L_c^{\vartheta}$ to derive that

$$\mathscr{D}((M_k^\vartheta)^\omega)=\mathscr{D}((L_c^\vartheta)^\omega), \quad 0\leq\omega\leq 1$$

with equivalent norms. But, $(M_k^\vartheta)^\omega=M_k^{\vartheta\omega}$ and $(L_c^\vartheta)^\omega=L_c^{\vartheta\omega}$ since $\vartheta<1$ [5]. This shows that

$$\mathscr{D}(M^\omega)=\mathscr{D}(M_k^\omega)=\mathscr{D}(L_c^\omega), \quad 0\leq\omega\leq\vartheta$$

with equivalent norms. Q.E.D.

Details of the proof and related control theoretic results will appear elsewhere.

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