74. A Free Boundary Problem for Nonlinear Elliptic Equations

By Seiro OMATA*) and Yoshihiko YAMAURA**)

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 9, 1990)

Abstract: In this paper, we treat a free boundary problem for nonlinear elliptic equations derived from a variational problem. Linear and quasilinear cases of this problem have been studied by H. W. Alt, L. A. Caffarelli and A. Friedman. We treat nonlinear case and show that the free boundary is a regular curve, when the domain is two dimensional, under a rather strong one sided condition for the coefficient of the functional.

1. Introduction. There are some results obtained by H. W. Alt, L. A. Caffarelli and A. Friedman about functionals with a variable boundary. (See [1] and [2].)

Their problems are as follows: for u: $\Omega\to R$, $\Omega\subset R^n$, consider the functional:

$$I(u)=\int_\Omega (F(|\nabla u|^2)+Q^2(x)\chi_{u>0})dL^n,$$

where L^n is n dimensional Lebesgue measure and $Q(x)$ is a given measurable function with $0<Q_{\min}\leq Q(x)\leq Q_{\max}$ and χ denotes a characteristic function and $\Omega(\subset R^n)$ is an open and connected domain (may be unbounded) with Lipschitz boundary. Here and in the sequel we denote $\{x\in\Omega\,;\,u(x)>0\}=\Omega(u>0)$, and $\chi_{u>0}$ is the function of the set $\Omega(u>0)$. In [1], the case $F(t)=t$ was treated, and in [2], the case $F(t)$ belonging to $C^{2,1}[0,\infty)$, with $F(0)=0$ and $0<c\leq(\partial F/\partial t)\leq C$ and $0\leq((1/1+t)(\partial^2F/\partial t^2))\leq C$. It was proved that if $Q(x)$ is Hölder continuous, roughly speaking, the free boundary $\Omega\cap\partial\Omega(u>0)$ is a $C^{1,\beta}$-curve in any compact subset of Ω, provided that u is a minimizer of I. These results are applied to solve the Jet problem and the cavitational flow problem (see [3–7] and [11]).

We extend these result ([1] and [2]) to the following nonlinear problem. Consider the minimizing problem:

$$J(u)=\int_\Omega (a^{ij}(u)D_iuD_ju+Q^2\chi_{u>0})dL^n,$$

under the same assumption for χ and Ω as in [1] and here Q is assumed to be a positive constant. (We used summation convention.) We need some further assumption for the coefficients $a^{ij}(z)$: $a^{ij}(z)$ belongs to class C^∞ with respect to z, and satisfies the following ellipticity and boundedness conditions, $0<\lambda|\xi|^2\leq a^{ij}(z)\xi_i\xi_j\leq\Lambda|\xi|^2$ for all $\xi\in R^n-\{0\}$, moreover $[\dot a^{ij}(z)]$, the derivative of $[a^{ij}(z)]$ with respect to z, is positive definite. We call

*) Kitami Institute of Technology.
**) Department of Mathematics, Keio University.

this the strong one sided condition.

Under these assumptions, we find a minimizer in the function set K, where $K=\{u \in L^2_{\mathrm{loc}}(\Omega) \mid \nabla u \in L^2(\Omega),\ u=u^0 \text{ on } S\}$. Here u^0 is a given function with $u^0 \in L^2_{\mathrm{loc}}(\Omega)$, $\nabla u^0 \in L^2(\Omega)$, and $0 \leq u^0 \leq \sup_\Omega u^0 < +\infty$, and S is a subset of $\partial\Omega$ with a positive $n-1$ dimensional Hausdorff measure.

We show that if Ω is 2 dimensional, the free boundary of the minimizer J is a $C^{1,\beta}$ curve in any compact subset Ω.

2. Regularity of a minimizer. The existence theorem is a direct conclusion of the lower semicontinuity of the functional J under an assumption $J(u^0)<\infty$ (see [1]). The boundedness of a minimizer is obtained in the same way as in [1], using the test function $u+\min(\sup_\Omega u^0-u, 0)$ and $u-\min(u, 0)$.

We can treat u by the method of Ladyzhenskaya and Ural'tseva, and obtain the Hölder continuity of a minimizer.

Theorem 2.1. *If u is the minimizer, then there exists $\alpha>0$ depending on $\tilde{\Omega}$, such that $u \in C^\alpha(\tilde{\Omega})$, where $\tilde{\Omega}$ is a subdomain whose closure is compactly contained in Ω.*

By the theorem above, $\Omega(u>0)$ should be an open set, then u satisfies the following equation;

$$\int_{\Omega(u>0)} \left(-a^{ij}(u)D_i u D_j \varphi - \frac{1}{2}\dot{a}^{ij}(u)D_i u D_j u \varphi\right) dL^n = 0$$

for all $\zeta \in C_0^\infty(\{u>0\})$. (In the sequel, we denote left hand side Lu.) By using this equation, the higher regularity can be easily obtained (see [12] and [14]). In other words, $u \in C^\infty(\Omega(u>0))$.

Since $0 \leq J(u-\varepsilon\zeta)-J(u)$ for arbitrary $\zeta \in C_0^\infty(\Omega)$, $\zeta \geq 0$ and $\varepsilon>0$, we have the following differential inequality in the whole domain Ω:

$$\int_\Omega \left(-a^{ij}(u)D_i u D_j \varphi - \frac{1}{2}\dot{a}^{ij}(u)D_i u D_j u \varphi\right) dL^n \geq 0.$$

From this equation, we cannot obtain further regularity results by using usual methods.

Next we will consider Lipschitz continuity and nondegeneracy of the minimizer. To obtain the Lipschitz continuity, which plays an essential part for constructing a Radon measure, we should use the method of Alt-Caffarelli-Friedman (see [2]).

Theorem 2.2. *Let u be a minimizer, and choose $x_0 \in \Omega$ arbitrary with* $\operatorname{dist}(x_0, \Omega(u=0))<(1/2)\operatorname{dist}(x_0, \partial\Omega)$, *then there is constant $C=C(n, \lambda, \Lambda)$ such that*

$$u(x_0) \leq C \operatorname{dist}(x_0, \Omega(u=0)).$$

In the non-linear case, a proof of this theorem has a little difference from [2]. Essentially, we use iteration methods using the Morrey type estimate for the Hölder continuous function. (See [2] and [15].)

By using the Lipschitz continuity of the miminum, we have a nondegeneracy theorem.

Theorem 2.3. *For any $p>1$ and for any $0<\kappa<1$, there is a constant $C_\kappa=C(n,\kappa)$, such that for any balls B_r with radius r contained in Ω,*

$$\frac{1}{r}\left(\frac{1}{|B_r|}\int_{Br} u^p\right)^{1/p} dx \le C_\kappa$$

implies $u=0$ in $B_{\kappa r}$, provided that u is a minimizer.

3. Identification of the differential q_u. Our aim of this paper is now to prove that the free boundary of a minimizer, $\partial\Omega(u>0)=\Omega\cap\partial\{x\in\Omega;\ u(x)>0\}$, becomes locally the graph of a $C^{1,\alpha}$-function ($\alpha\in(0,1)$). First, we will show that $\partial\Omega(u>0)$ is an $(n-1)$-dimensional surface in some weak sense. (See [16].) For this, we will introduce the following Radon measure:

$$\lambda(D)=\sup_{\varphi\in C_0^1(D),|\varphi|\le 1}\int_D\left(-a^{ij}(u)D_iuD_j\varphi-\frac{1}{2}\dot{a}^{n}(u)D_iuD_juu\varphi\right)dL^n,$$

where D is an arbitrary open set, which is compactly contained in Ω. On this Radon measure λ, the following fact is proved in [2] and [15]: For any Borel measurable set $E\subset\partial\Omega(u>0)\cap D$

$$cH^{n-1}(E)\le\int_E d\lambda\le CH^{n-1}(E), \tag{3.1}$$

where c and C depend only on D. In particular the left inequality of (3.1) indicates the local finiteness of the free boundary with respect to the $n-1$ dimensional Hausdorff measure. From this fact, we can conclude that the free boundary $\partial\Omega(u>0)$ is the $(n-1)$-dimensional surface with locally finite perimeter in Ω. (See [9].) Moreover (3.1) shows that the Radon measure λ is absolutely continuous with respect to $H^{n-1}\lfloor\partial\Omega(u>0)$. Thus we get the following representation:

$$\int_\Omega\left(-a^{ij}(u)D_iuD_j\varphi-\frac{1}{2}\dot{a}^{ij}(u)D_iuD_juu\varphi\right)dL^n=\int_{\partial\Omega(u>0)}\varphi q_u dH^{n-1}$$
$$\text{for all } \varphi\in C_0^\infty(\Omega)$$

where

$$q_u(x)=\lim_{\rho\to 0}\frac{\lambda(B_\rho(x))}{H^{n-1}(B_\rho(x)\cap\partial\Omega(u>0))} \qquad (x\in\partial\Omega(u>0)).$$

Now we introduce the blow up of the minimum u:

$$u_{m,x_0}(x)=\frac{1}{\rho_m}u(x_0+\rho_m x) \qquad (\rho_m\longrightarrow 0).$$

Without loss of generality, by an adequate change of coordinates, we can assume $a^{ij}(0)=\delta^{ij}a(0)$. We can show that the blow up limit u_{x_0} achieves the minimum of the following functional which is related to the Laplace-equation:

$$I(w)=\int_{B_R(0)}\left(|\nabla w|^2+\frac{Q^2}{a(0)}\chi_{w>0}\right)dL^n. \tag{3.2}$$

Moreover, for a.e. $x_0\in\partial\Omega$, the blow up limit u_{x_0} is represented by a following linear function:

$$u_{x_0}(x)=\frac{q_u(x_0)}{a(0)}\max(\langle x, \nu_u(x_0)\rangle, 0).$$

Thus we get the next equality, or so-called identification:

$$q_u \equiv \sqrt{a(0)}Q \quad \text{a.e.} \quad \partial\Omega(u>0).$$

4. Blow up limit of a minimizer ($n=2$). In this section, we will mention the blow up limit in the special case $n=2$. Since the blow up limit of a minimizer u_{x_0} is represented as (3.2), we can proceed in the same way as in [1].

As the first step we get the next equality using the notion of the blow up limit:

$$\varlimsup_{x\to x_0,\, u(x)>0} |\nabla u(x)| = \frac{Q}{\sqrt{a(0)}} \tag{4.1}$$

for $x_0 \in \partial\Omega(u>0)$.

Secondly we obtain the following estimate which holds only in the case $n=2$:

$$\frac{1}{L^n(B_r(u>0))}\int_{Br(u>0)}\left(\frac{Q^2}{a(0)}-|\nabla u|^2\right)^+ \leq \frac{C}{\log\frac{1}{r}}, \tag{4.2}$$

where u is the minimizer of the functional J and B_r is a sufficiently small n-dimensional ball contained in Ω with the center on the free boundary.

From (4.1) and (4.2), only in the case $n=2$ we conclude that for all $x_0 \in \partial\Omega(u>0)$, the blow up limit of a minimizer u_{x_0} is the half plane solution.

5. Regularity of the free boundary. We can show that all free boundary points have their normal vector a.e. H^{n-1}. In this section, we will show the Hölder continuity of the normal vector of the free boundary. The notion of non-homogeneous blow up plays an essential role of this proof. (See [1–2].) Here, we need some definitions for non-homogeneous blow up.

Definition 5.1. Let σ_0, $\sigma_+ \in (0,1]$ and $\tau>0$. We say that the minimum u belongs to $F(\sigma_0, \sigma_+; \tau)$ in $B_\rho(0)$ with respect to e_n, if u satisfies following conditions.

$$\begin{aligned}
&u(x)=0 && \text{in } B_\rho(x_n\geq\sigma_0\rho),\\
&u(x)\geq\frac{Q}{\sqrt{a(0)}}((-x_n)-\sigma_-\rho) && \text{in } B_\rho(x_n<-\sigma_+\rho),\\
&|\nabla u|\leq\frac{Q}{\sqrt{a(0)}}(1+\tau) && \text{in } B_\rho.
\end{aligned}$$

Using the method in [2], we obtain the following theorem, an improvement of the plus flatness condition.

Theorem 5.2. *Let $\rho<1$ and $\sigma<\min(1/10, \sigma_0(n, \lambda, M))$, and suppose $\rho<\sigma$, then there exists $C=C(n, \lambda, M)$ such that $u\in F(\sigma, 1; \sigma)$ in B_ρ w.r.t. ν implies $u\in F(2\sigma, C\sigma; \sigma)$ in $B_{\rho/2}$ w.r.t. ν.*

Definition 5.3 (Non-homogeneous blow up). Let $u_k\in F(\sigma_k, \sigma_k; \tau_k)$ in $B_{\rho_k}(y_k)$, where $\{\sigma_k\}$ is a sequence which is chosen $\sigma_k\to 0$ as $n\to\infty$ and $\rho_k<\sigma_k$

for all k and $\tau_k=O(\sigma_k)$. Then we define

$$f_k^+(\bar{x})=\sup\{x_n\,|\,(\rho_k\bar{x},\ \sigma_k\rho_k x_n)\in\partial\{u_k>0\}\},$$
$$f_k^-(\bar{x})=\inf\{x_n\,|\,(\rho_k\bar{x},\ \sigma_k\rho_k x_n)\in\partial\{u_k>0\}\}.$$

Using Theorem 5.2, it is easy to see that there is a subsequence such that

$$f:=\limsup_{\substack{j\to 0\\ z\to\bar{x}}} f_j^+(z)=\liminf_{\substack{j\to 0\\ z\to\bar{x}}} f_j^-(z).$$

Using f defined above, we can show the following lemma which is essential for the improvement of zero flatness condition.

Lemma 5.4. *Let u_k be the sequence of non-homogeneous blow up which satisfies the following conditions:*

$$u_k\in F(\sigma_k,\sigma_k\,;\tau_k)\qquad in\ B_{\rho_k}(x_k)\ w.r.t.\ \nu_k$$

and

$$\rho_k=o(\sigma_k),\qquad \tau_k=o(\sigma_k^2)$$
$$\int_0^{1/4}\frac{1}{r^2}\Big[Av_rf(\bar{x})-f(\bar{x})\Big]dr\le C\qquad(\bar{x}\in\mathcal{B}_{1/4}(0))$$

where $\mathcal{B}_r$ is a $n-1$ dimensional ball and $Av_rf(\bar{x})$ is the average of the integration of f on $\partial\mathcal{B}_r(\bar{x})$.

Combining Theorem 5.2 and Lemma 5.4, we can easily obtain the following lemmata.

Lemma 5.5 (Lipschitz continuity of the function f).

$$f\in C^{0,1}(\mathcal{B}_{1/4}(0))$$

Lemma 5.6 (Estimate for f from above by linear function). *For all $\theta>0$, there exists a positive number c_θ such that*

$$f(\bar{x})\le l\cdot\bar{x}+\frac{1}{2}\theta r\qquad \bar{x}\in\mathcal{B}_r(0)$$

for some $r\in[c_\theta,\theta]$ and l is the vector in R^{n-1} with $|l|\le c(n)$.

Using above lemmata, we immediately obtain the next lemma, an improvement of zero flatness condition.

Lemma 5.7 (Improvement of zero flatness conditions). *For all $\theta>0$, there exists a positive number c_θ and σ_θ such that if $u\in F(\sigma,\sigma\,;\tau)$ in B_ρ w.r.t. ν, (for $\forall\sigma\le\sigma_\theta$, $\forall\tau\le\sigma_\theta\sigma^2$, $\forall\rho\le c(n)\tau^{1/2}$), then $u\in F(\theta\sigma,1\,;\tau)$ in $B_{\bar\rho}$ w.r.t. $\bar\nu$ (for some $\bar{\rho}\in[c_\theta\rho,\theta\rho]$, $\bar{\nu}$ with $|\bar{\nu}-\nu|\le c(n)\sigma$).*

Using the iteration method, we obtain the theorem.

Theorem 5.8 (Improvement of all flatness conditions). *For all $\theta>0$, there exists a positive number c_θ and σ_θ such that if $u\in F(\sigma,1\,;\tau)$ in B_ρ w.r.t. ν, (for $\forall\sigma\le\sigma_\theta$, $\forall\tau\le\sigma_\theta\sigma^2$, $\forall\rho\le c(n)\tau^{1/2}$), then $u\in F(\theta\sigma,\theta\sigma,\theta^2\tau)$ in $B_{\bar\rho}$ w.r.t. $\bar\nu$ (for some $\bar{\rho}\in[c_\theta\rho,(1/4)\rho]$, $\bar{\nu}$ with $|\bar{\nu}-\nu|\le c(n)\sigma$).*

Finally we can show the conclusion of this paper, by using Theorem 5.8 and the well-known method by Federer ([8]).

Theorem 5.9 (Regularity of the free boundary). *Let D be an arbitrarily fixed subdomain compactly contained in Ω, then there exists a positive number $\sigma_0(n,\alpha)>0$, such that $u\in F(\sigma,1\,;\infty)$ in $B_\rho(x_0)\subset D$ w.r.t. ν, ($\forall\sigma\le\sigma_0$ and $\forall\rho\le\sigma_0\sigma^{2/\alpha}$) implies that there exists positive number $\nu(x_0)$, $\beta=\beta(n)$,*

$C=C(n)$ *such that*

$$|\langle(x-x_0), \nu(x_0)\rangle|\leq\frac{C\sigma}{\rho^\beta}|x-x_0|^{1+\beta} \qquad (x\in B_{\rho/2}(x_0)\cap\partial\{u>0\}).$$

From this, it immediately follows that the free boundary is a $C^{1,\beta}$ surface.

6. Acknowledgments. We greatly thank Professor N. Kikuchi (Keio University) and Professor L.A. Caffarelli (Institute for Advanced Study, Princeton) for their good advices.

References

[1] H. W. Alt and L. A. Caffarelli: Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math., **325**, 105–144 (1981).
[2] H. W. Alt, L. A. Caffarelli and A. Friedman: A free boundary problem for quasilinear elliptic equations. Ann. Scuola Norm. Sup. Pisa, (4) **11**, 1–44 (1984).
[3] ——: Asymmetric jet flows. Comm. Pure Appl. Math., **35**, 29–68 (1982).
[4] ——: Jet flow with gravity. J. Reine Angew. Math., **331**, 58–103 (1982).
[5] ——: Variational problem with two phases and their free boundaries. T. A. M. S., **283**, no. 2, 431–460 (1984).
[6] L. A. Caffarelli: A Harnack inequality approach to regularity of free boundaries part I. Revista Matematica Iberoamericana, **3**, no. 2, 139–162 (1987).
[7] ——: A Harnack inequality approach to regularity of free boundaries part II. Flat free boundaries are Lipschitz. Comm. Pure Appl. Math., XLII, 55–78 (1989).
[8] H. Federer: Geometric Measure Theory. Springer-Verlag, **153**, Berlin-Heidelberg-New York (1977).
[9] E. Giusti: Minimal Surfaces and Functions of Bounded Variation. Birkhäuser (1984).
[10] M. Giaquinta: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Princeton University Press (1983).
[11] ——: Topics in Caluculus of Variations. Lect. Notes in Math., **1365**, Springer-Verlag (1989).
[12] D. Gilbarg and N. S. Trudinger: Elliptic Partial Differential Equations of Second Order. 2nd ed., Springer-Verlag (1984).
[13] M. Grüter and K-O. Widman: The green function for uniformly elliptic equations. Manuscripta Math., **37**, 303–342 (1982).
[14] O. A. Ladyzhenskaya and N. N. Ural'tseva: Linear and Quasilinear Elliptic Equations. Academic Press (1968).
[15] S. Omata: A Lipschitz estimates for a free boundary problem for nonlinear elliptic equations (preprint).
[16] L. Simon: Lecture on geometric measure theory. Proceedings of the Centre for Mathematical Analysis Australian National University, vol. 3 (1983).