70. An Example of an Exceptional $(-2n-1, n)$-curve in an Algebraic 3-fold

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§1. **Introduction.** Let X be a non-singular algebraic variety of dimension three over the complex number field C. A non-singular curve $C\subset X$ is called *exceptional* if there exists a variety Y and a morphism $f: X\to Y$ such that $f(C)$ is a point and that $X-C\cong Y-f(C)$. A non-singular rational curve $C\subset X$ is called *an* (a, b)-*curve*, if the normal bundle of C can be presented as $N_{C/X}=\mathcal{O}_C(a)\oplus\mathcal{O}_C(b)$ with $a\leqq b$. Let C be an exceptional (a, b)-curve. What value can the pair (a, b) take? By Ando[1] and Nakayama[3], the following facts are proved.

(1) (a, b) must satisfy an inequality $a+2b\leqq -1$ and $(a, b)\neq(-1, 0)$.

(2) For every pair (a, b) with $a+2b\leqq -2$, there exist some examples of exceptional (a, b)-curves.

Then we have a question: For (a, b) with $a+2b=-1$ and $(a, b)\neq(-1, 0)$, does there exist an example of an exceptional (a, b)-curve?

For $(a, b)=(-3, 1)$, some examples are constructed and properties of $(-3, 1)$-curves are studies in Laufer[2], Pinkham[4], Reid[5] and Ando[1]. But for $b\geqq 2$, no examples are known yet. In this paper, we shall show that for every pair (a, b) with $a+2b=-1$ and $(a, b)\neq(-1, 0)$, there exists an example of an exceptional (a, b)-curve.

§2. **Construction.** Put $(a, b)=(-2n-1, n)$ with $n>0$. We shall construct a $(-2n-1, n)$-curve C in an algebraic 3-fold X. (Compare with Laufer[2].)

Let U and V be C^3 with coordinates (x, y_1, y_2) and (w, z_1, z_2). We construct X by patching U and V by the following transition functions:

$$\begin{cases} z_1=x^{2n+1}y_1+y_2^2+x^n y_2^3 \\ z_2=x^{-n}y_2 \\ w=x^{-1} \end{cases}$$

It is clear that the curve $C\subset X$ defined by $y_1=y_2=0$ in U and $z_1=z_2=0$ in V is a rational curve with $I_C/I_C^2\cong\mathcal{O}_C(2n+1)\oplus\mathcal{O}_C(-n)$ where I_C is a defining ideal of C in $\mathcal{O}_X$. We shall show that this $(-2n-1, n)$-curve C is exceptional. The first step is to find five holomorphic functions $h_1, \cdots, h_5$ on X. Let $h_1=z_1=x^{2n+1}y_1+y_2^2+x^n y_2^3$ and $h_2=w^{2n}z_1-z_2^2=xy_1+y_2^3$. h_1 and h_2 are holomorphic functions on X. Since $u=h_2^2-h_1^3=x(2y_1y_2^3+y_1^2)+x^{2n}u_0(x, y_1, y_2)$ can be divided by x, $h_3=z_2u^n$ and $h_4=wu$ are also holomorphic functions on X. For $r\geqq 0$, let

$$f_{0,r}=\begin{cases} w^n z_1^{r/2} & \text{if } r \text{ is even} \\ z_2 z_1^{(r-1)/2} & \text{if } r \text{ is odd.} \end{cases}$$

Inductively, for $q\geqq 1$, let $f_{q,r}=f_{0,r}h_2^q-\sum_{i=0}^{q-1}\binom{q}{i}f_{i,3q-3i+r}$.

Lemma. $\sum_{i=a}^{b}(-1)^{i-a}\binom{b}{i}\binom{i}{a}=0.$

Proof. Let $r=b-a$. Then

$$\begin{aligned}\sum_{i=a}^{b}(-1)^{i-a}\binom{b}{i}\binom{i}{a}&=\frac{b\,!}{a\,!}\sum_{i=a}^{b}\frac{(-1)^{i-a}}{(b-i)\,!(i-a)\,!}\\&=\frac{b\,!}{a\,!}\sum_{j=0}^{r}\frac{(-1)^j}{(r-j)\,!j\,!}\\&=\frac{b\,!}{a\,!r\,!}\sum_{j=0}^{r}(-1)^j\binom{r}{j}=0.\end{aligned}$$

Let $t_i^{(q,r)}=\begin{cases} w^n z_1^{(3q+r-3i)/2}h_2^i & \text{if } q+r+i \text{ is even} \\ z_2 z_1^{(3q+r-3i-1)/2}h_2^i & \text{if } q+r+i \text{ is odd.} \end{cases}$

Claim. $f_{q,r}=\sum_{i=0}^{q}(-1)^{q-i}\binom{q}{i}t_i^{(q,r)}.$

Proof. Note that $t_i^{(k,3q-3k+r)}=t_i^{(q,r)}$. Let $t_i=t_i^{(q,r)}$. We shall show that $f_{k,3q-3k+r}=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}t_i$. Note that when we put $k=q$, we have the claim. If $k=0$, the above equality trivially holds. If $k>0$, we shall prove this by the induction on k. Since $f_{0,3q-3k+r}h_2^k=t_k^{(q,r)}$,

$$\begin{aligned}f_{k,3q-3k+r}&=f_{0,3q-3k+r}h_2^k-\sum_{i=0}^{k-1}\binom{k}{i}f_{i,3k-3i+r}\\&=t_k-\sum_{i=0}^{k-1}\binom{k}{i}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}t_j\\&=t_k-\sum_{j=0}^{k-1}\sum_{i=j}^{k-1}(-1)^{i-j}\binom{k}{i}\binom{i}{j}t_j\\&=t_k+\sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}t_j\\&=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}t_j.\end{aligned}$$

Thus we complete the proof.

Trivially, $f_{q,r}$ is a polynomial with respect to w, z_1, z_2. On the other hand by construction, $f_{q,r}(x,y_1,y_2)=x^{q-n}y_1^q y_2^r+x^n g(x,y_1,y_2)$, where g is a suitable polynomial. Especially, $f_{n,0}$ is also a polynomial with respect to x, y_1, y_2. Thus, $h_5=f_{n,0}$ is a holomorphic function on X.

Now we have a holomorphic mapping $h=(h_1,\cdots,h_5)\colon X\to C^5$. Let $X\xrightarrow{f}Y\xrightarrow{g}C^5$ be the Stein factorization of h. Since $C=h^{-1}(0)$, $f(C)$ is a point. We shall show that $X-C\cong Y-f(C)$. Since g is a finite map, and since every fiber of f is connected, it is enough to show that h is finite except the origin 0.

Let $v=(v_1,\cdots,v_5)\in h(V)-\{0\}$. If $v_2^2-v_1^3\neq 0$, then $z_1=v_1$, $z_2=v_3/(v_2^2-v_1^3)^n$ and $w=v_4/(v_2^2-v_1^3)$. Thus $h^{-1}(v)$ is just a point.

Assume $v_2^2-v_1^3=0$. If $v_2=0$, then we can derive $v_1=v_2=v_3=v_4=v_5=0$.

Thus $v_2 \neq 0$. Since $h_5 = f_{n,0} = v_5$, we have $\alpha w^n - \beta z_2 = v_5$, where

$$\alpha = \begin{cases} \sum_{j=0}^{n/2} \binom{n}{2j} v_1^{(3n-6j)/2} v_2^{2j} & \text{if } n \text{ is even} \\ \sum_{j=0}^{(n-1)/2} \binom{n}{2j+1} v_1^{(3n-6j-3)/2} v_2^{2j+1} & \text{if } n \text{ is odd} \end{cases}$$

and

$$\beta = \begin{cases} \sum_{j=0}^{(n/2)-1} \binom{n}{2j+1} v_1^{(3n-6j-4)/2} v_2^{2j+1} & \text{if } n \text{ is even} \\ \sum_{j=0}^{(n-1)/2} \binom{n}{2j} v_1^{(3n-6j-1)/2} v_2^{2j} & \text{if } n \text{ is odd.} \end{cases}$$

Since $v_2^2 = v_1^3$ and since $v_2 \neq 0$, we have $(\alpha, \beta) \neq (0, 0)$. Thus the equation on w and z_2

$$\begin{cases} \alpha w^n - \beta z_2 = v_5 \\ v_1 w^{2n} - z_2^2 = v_2 \end{cases}$$

have at most $2n$ common solutions. Thus $h|_{V-C}$ is finite.

Let $v = (v_1, \cdots, v_5) \in h(U-V) - \{0\}$. Then $w=0$. Thus $v_1 = y_2^2$, $v_2 = y_2^3$, $v_3 = 2^n y_1^n y_2^{3n+1}$, $v_4 = 2y_1 y_2^3$, and $v_5 = y_1^n$. Therefore $h|_{U-V-C}$ is finite.

Thus we have $h|_{X-C}$ is finite. Therefore C is exceptional.

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