7. *A Note on the Problem of Yokoi*

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Let p be a prime congruent to 1 mod 4 and $\varepsilon_p=(t+u\sqrt{p})/2>1$ be the fundamental unit of $Q(\sqrt{p})$. From Theorem 1 of [1], there exist only a finite number of real quadratic fields $Q(\sqrt{p})$ with class number one for any fixed positive integer u. The problem of enumerating these fields for the cases $u=1$ and $u=2$ was solved by H. K. Kim, M.-G. Leu and T. Ono ([2]).

In this paper, we shall determine all these fields for $1\leqq u\leqq 300$ in proving the following theorem.

Theorem. *With the above notation, there exist at most* 44 *real quadratic fields* $Q(\sqrt{p})$ *with class number one for* $1\leqq u\leqq 300$, *where* p *are those in Table* II *with one possible exception.*

Proof. Let χ_p be the Kronecker character belonging to $Q(\sqrt{p})$ and $L(s,\chi_p)$ be the corresponding L-series. Then by Theorem 2 of [4], for any $y\geqq 12$, we have

$$L(1,\chi_p)>\frac{0.655}{y}p^{-1/y}$$

with one possible exception of p, where $y=\log p$.

Further, from class number formula, for any $e^y\leqq p$ $(y\geqq 12)$, we have

$$\begin{aligned} h(p)&=\frac{\sqrt{p}}{2\log\varepsilon_p}L(1,\chi_p)\\ &>\frac{0.655}{y}\frac{\sqrt{p}\,p^{-1/y}}{2\log(u\sqrt{p})}=\frac{0.655}{y}\frac{p^{(y-2)/2y}}{2\log u+\log p}\\ &\geqq\frac{0.655e^{(y-2)/2}}{y(y+2\log u)}.\end{aligned}$$

Thus $h(p)=1$ implies

$$(1)\qquad 0.655e^{(y-2)/2}\leqq y(y+2\log u).$$

Put for convenience

$$g(x,y)=\frac{0.655e^{(y/2)-1}}{y(y+2x)},\ \text{where } x=\log u.$$

The curve C in Figure 1 represents the graph of $g(x,y)=1$. The inequality (1) means that the point $(\log u, \log p)$ with $h(p)=1$ should lie in the shadowed domain in this figure. In particular, $1\leqq u\leqq 2$ implies $1\leqq p\leqq e^{14}$ and $5\leqq u\leqq 300$ implies $1\leqq p\leqq e^{15}$.

Now put

$$U=\{2^r\prod p_i^{s_i}\mid r=0\text{ or }1,\ p_i\equiv 1\ (\text{mod }4),\ s_i\geqq 0\}.$$

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Table I

u	p	n	q	h
1	5	1	5	1
	13	3	3	1
	29	5	5	1
	53	7	7	1
	173	13	13	1
	293	17	17	1
2	17	1	5	1
	37	3	3	1
	101	5	5	1
	197	7	7	1
	677	13	13	1
5	61	2	3	1
	149	2	5	1
	317	4	7	1
	773	6	11	1
10	41	1	2	1
	269	2	5	1
	557	2	7	1
	1901	4	5	3
13	797	2	11	1
	2477	4	19	1
17	157	1	3	1
	461	1	5	1
25	109	0	3	1
	12197	4	11	3
26	45533	8	11	5
	100733	12	13	7
29	1013	1	11	1
	8069	3	5	3
	191693	15	17	7
34	1009	1	2	7
	4933	2	3	3
	42533	6	7	7
37	941	1	5	1
	1877	1	7	1
	10957	3	3	7
41	509	1	5	1
	3533	1	13	1
	220373	11	13	11
50	8597	2	7	3
58	761	0	2	3
	7817	2	2	5
	40637	3	7	5
61	1493	1	7	1
	6949	1	3	5
	163733	7	11	7
65	653	0	7	1
	3221	1	5	3
	10909	2	3	5
	24197	2	13	3
	34877	3	23	3
	111053	5	17	5
73	2677	1	3	3
	1983197	19	37	15
74	1613	1	13	1
	13033	2	2	13

u	p	n	q	h
82	5297	1	2	3
	23957	2	17	3
	113717	4	7	7
	442517	8	11	15
85	2693	1	11	1
	52301	3	5	7
	79133	3	17	5
	209861	5	5	19
	1148957	13	13	15
89	30893	2	7	5
	282773	6	7	11
97	181	0	3	1
	43037	2	7	5
101	85733	3	7	7
105	89	0	2	1
	49033	2	2	21
	95213	3	13	7
	564653	7	11	17
109	49109	2	5	9
113	95317	3	3	9
	2247797	13	29	19
122	12281	1	2	11
	17737	1	2	15
	552317	6	13	13
125	153733	3	3	13
	369557	5	11	9
	412277	5	23	9
	736013	7	11	15
130	389	0	5	1
	12161	1	2	11
	17989	1	3	9
	69761	2	2	21
	155333	3	13	7
	427877	5	7	13
	2372453	12	17	19
137	1637	0	11	1
	54541	2	3	17
	743837	6	13	13
145	12269	1	5	3
	1005677	7	7	19
146	113	0	2	1
	24533	1	11	3
	1698773	9	23	17
149	15773	1	7	3
	179429	3	5	9
157	277	0	3	1
	373757	4	29	7
170	10313	1	2	7
	56857	1	2	29
	124717	2	3	9
	480461	4	5	21
	905453	6	13	15
173	397	0	3	1
	506837	4	7	19
	3062237	10	13	21
178	2081	0	2	5
	96337	2	2	35
	335957	3	31	7

u	p	n	q	h
181	20389	1	3	5
	1703573	7	7	25
185	1277	0	11	1
	111733	2	3	11
	164621	2	5	9
	971141	5	5	19
	1524077	7	7	25
193	399221	3	5	23
	16857917	21	43	35
194	23173	1	3	5
	55589	1	5	7
	3196133	9	31	15
	4377413	11	13	29
197	1862837	7	7	27
202	11257	1	2	17
205	1997	0	7	1
	1383653	6	19	13
	1647917	6	19	15
218	2777	0	2	3
	37277	1	7	5
	146921	2	2	21
221	31973	1	7	3
	62213	1	7	5
	478069	3	3	45
	2264453	7	7	33
	6048773	11	11	43
226	218213	2	7	9
229	692333	4	11	15
233	215077	2	3	21
250	73	0	2	1
	258617	2	2	23
257	74597	1	13	5
	3294677	7	7	53
265	92189	1	5	7
	707573	3	11	15
	1880117	5	7	21
	3299213	7	71	35
269	6362453	9	11	39
274	7873	0	2	9
	210929	2	2	33
	2127893	5	11	23
	26186453	19	19	63
277	2557	0	3	3
	1342277	4	13	15
281	62189	1	5	7
	3994493	7	13	29
289	9749	0	5	3
	229693	2	3	19
290	94321	1	2	37
	356549	2	5	17
	1306181	4	5	27
298	137	0	2	1
	95917	1	3	9

Table II

	293												
	173	677											
	53	197	773										
p	29	101	317	557									
	13	37	149	269	2477	461			1877	3533			
	5	17	61	41	797	157	109	1013	941	509	1493	653	1613
u	1	2	5	10	13	17	25	29	37	41	61	65	74

p	2693	181	89	389	1637	113	277	397	1277	1997	73	137
u	85	97	106	130	137	146	157	173	185	205	250	298

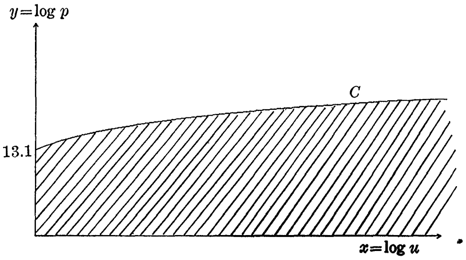

Fig. 1

It is also known that $u \in U$ and p is expressed in the form $p=u^2n^2\pm 2an+b$, where a, b, n are integers such that $0\leqq a<u^2/2$, $a^2+4=bu^2$.

Let q be the least prime which satisfies $\chi_p(q)=1$. Then it is also known that $h(p)\geqq \log n/\log q$ (cf. [3]). Therefore if $h(p)=1$, then $q\geqq n$ holds.

Table I lists all u, p, q, n for $u \in U$, $1\leqq u\leqq 300$ and $h=h(p)$ calculated by computer. Table II gives all p's with $h(p)=1$.

Remark. As above mentioned for each $u\in U$, p is expressed in the form $u^2n^2\pm 2an+b$ ($a^2+4=bu^2$). Since $(u^2, 2a, b)=1$, $u^2\pm 2a$ and b are not both even, and $(2a)^2-4u^2b=-16$ is not a square, p satisfies the conditions of Hardy-Littlewood Conjecture (F) ([5]). Therefore, if we assume the Conjecture (F), there should exist infinitely many primes $p=u^2n^2\pm 2an+b$ for each u. But there exist only finitely many real quadratic fields $\boldsymbol{Q}(\sqrt{p})$ with class number one for such p's. Hence, roughly speaking, the set of the real quadratic fields $\boldsymbol{Q}(\sqrt{p})$ with class number one is in a sense very small in the set of all the real quadratic fields $\boldsymbol{Q}(\sqrt{p})$ (prime $p\equiv 1$ (4)).

References

[1] H. Yokoi: Class number one problem for real quadratic fields. Proc. Japan Acad., **64**A, 53–55 (1988).

[2] H. K. Kim, M. -G. Leu, and T. Ono: On two conjectures on real quadratic fields. ibid., **63**A, 222–224 (1987).

[3] H. Yokoi: Some relations among new invariants of prime number p congruent to 1 mod 4. Adv. Studies in Pure Math., **13**, 493–501 (1988).

[4] T. Tatuzawa: On a theorem of Siegel. Japanese J. Math., **21**, 163–178 (1951).

[5] G. H. Hardy and J. E. Littlewood: Some problems of Partitio numerorum. III. On the expression of a number as a sum of primes. Acta Math., **44**, 1–70 (1923).