4. A Remark on Free Boundary Plateau Problem for Surfaces of General Topological Type*)

By Nobumitsu NAKAUCHI
Department of Mathematics, Faculty of Science, Yamaguchi University

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The classical Plateau problem asks for a disk type surface of least area spanning a prescribed Jordan curve in R^3. It was independently solved by Douglas and Radó in 1931. Douglas [3] considered also the case for surfaces of general topological type or surfaces of any genus with any number of boundary components, and solved it under a so called "Douglas' condition". Courant derived a different proof using not Douglas functional, but Dirichlet integral. (See Courant [2].) He also gave another notion or "the condition of cohesion", which guarantees the existence of solutions. Courant's proof was restricted to the case of surfaces of genus 0, but Shiffman [8] proved it for general case. Recently Tomi-Tromba [9] presented a modern approach to Shiffman's result, using Teichmüller space.

In this short note, we remark that a free boundary version can be described along this line, under what we call "the condition of inter-border cohesion", in addition to the condition of cohesion. In the author's paper [6], we utilized "boundary incompressibility". This qualitative notion corresponds to the quantitative one or the condition of inter-border cohesion, while the incompressibility corresponds to the condition of cohesion.

Free boundary Plateau problem. Let M be a compact smooth surface with boundary components $C_1, \cdots, C_k$. Let S be a compact smooth submanifold of R^n, on which all our free boundaries should lie. We give here a class of "surfaces (or mappings) with free boundary on S" as follows:

$$\mathcal{F} := C^0(M, R^n) \cap W^{1,2}(M, R^n) \cap \{f\,;\, f(C_j) \subset S\},$$

where $W^{1,2}(M, R^n)$ denotes the ordinary Sobolev space of L^2-elements whose first derivatives (in the sense of distributions) belong also to L^2. We want to minimize the area

$$A(f) := \int_M \|df \wedge df\|,$$

in this class $\mathcal{F}$. However the infimum may be attained in general only by maps from the surfaces of lower topological types. So we recall the following condition for non-degeneracy (see Courant [2], p. 145, Tomi-Tromba [9], p. 51). This condition can be expressed, at least formally, in terms concerning only the present class, while Douglas' condition needs an information of the infimum of area in the class of all degenerate maps.

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Definition (the condition of cohesion). A sequence $\{f_n\}$ of $\mathfrak{F}$ satisfies *the condition of cohesion* if there exists a positive lower bound for the length of the images under any f_n of all homotopically non-trivial closed loops on M.

This "cohesion" prevents collapsings of mappings, and hence the existence of a minimizing sequence for the area satisfying the condition of cohesion guarantees the existence of solution for fixed boundary case. In free boundary case, there are other possibilities of degeneracy, since two free boundaries may approach close to each other, and the image by f_n may tends to 1-dim. complex as $n \to \infty$. We need other "cohesion" than the above one between boundary components. So we give here:

Definition (the condition of inter-border cohesion). A sequence $\{f_n\}$ of $\mathfrak{F}$ satisfies *the condition of inter-border cohesion* if there exists a positive lower bound for the length of the images under any f_n of all homotopically non-trivial arc relative to ∂M on M. Here an arc is called to be homotopically non-trivial relative to ∂M, if this arc can not be deformed continuously on M to a point under the restraint that two end points always lie on ∂M.

Under the above two conditions of cohesion, we have a free boundary version of Shiffman's result (cf. Tomi-Tromba [9], § 4):

Theorem. *If there is a minimizing sequence for the area in the class $\mathfrak{F}$ satisfying the conditions of cohesion and inter-border cohesion, then there exists a mapping of least area in this class.*

We will sketch our proof briefly. This free boundary problem can be reduced to fixed boundary problems for Jordan curves approaching S. (See Nakauchi [6].) Each fixed boundary area-minimizing problem is divided into two parts; a minimizing problem for Dirichlet's integral, and a variational problem in Teichmüller space. The first one is well-known, and can be solved by the standard argument. In the second part, the condition of inter-border cohesion implies that the boundary Jordan curves can not approach each other. Hence the two conditions of cohesion say that the length of any loop in the double of M is not less than a positive constant, i.e. the admissible conformal class is contained in a compact set of Teichmüller space. So we can find a minimizing solution.

Remarks. (1) In the above theorem, solutions are smooth up to boundary, since S is smooth. (cf. Jäger [5].) In case $n=3$, they have no interior branch point, i.e. they are immersions. (cf. Gulliver-Osserman-Royden [4].)

(2) We can show a similar existence result in the class:

$$\tilde{\mathfrak{F}} := W^{1,2}(M, R^n) \cap \{f\,;\ T(f) \in C^0(\partial M, S) \text{ as a } L^2\text{-element}\},$$

where $T: W^{1,2}(M, R^n) \to L^2(\partial M, R^n)$ be a trace operator. Note that an element of this class $\tilde{\mathfrak{F}}$ is not necessarily continuous, but the image of almost every loop and arc can be well-defined. (See Schoen-Yau [7], Asakura [1], Nakauchi [6].)

(3) The above results can be extended to partially free boundary problems. They remain valid also in case that the target may be a Riemannian manifold.

References

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