1. Theta Functions on the Classical Bounded Symmetric Domain of Type $\mathrm{I}_{2,2}$

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In this note, we announce a structure theorem on the graded ring of modular forms on the bounded symmetric domain of type $I_{2,2}$ with respect to the principal congruence subgroup of level $(1+i)$. Relations between the period map of certain K3 surfaces, the hypergeometric functions and the present theorem are studied in [8].

The classical bounded symmetric domain of type $I_{2,2}$ is defined by

$$\boldsymbol{D}:=\left\{W=(w_{jk})\ 1\leq j,k\leq 2 \,|\, \frac{W-W^*}{2i}>0\right\},\quad \text{where } W^*={}^t\overline{W}.$$

The group

$$U(2,2):=\left\{g\in GL(4,\boldsymbol{C})\,|\,g^*Jg=J,\ J=\begin{pmatrix}0 & I_2\\ -I_2 & 0\end{pmatrix}\right\},$$

acts on $\boldsymbol{D}$ by

$$g\cdot W=(AW+B)(CW+D)^{-1},\ \text{where } W\in\boldsymbol{D},\ g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in U(2,2),$$

and the transpose operator

$$T: W\to {}^tW,\qquad W\in\boldsymbol{D},$$

also acts on $\boldsymbol{D}$; these actions satisfy

$$(TgT)\cdot W=\bar{g}\cdot W.$$

Hence we have

$$\mathrm{Aut}(\boldsymbol{D})\simeq[U(2,2)/\{\text{center}\}]\rtimes\langle T\rangle,\qquad \langle T\rangle=\{id,T\}.$$

Let Γ be the modular group

$$\Gamma:=\{g\in GL(4,\boldsymbol{Z}[i])\,|\,g^*Jg=J\},$$

and let $\Gamma(1+i)$ be the congruence subgroup

$$\Gamma(1+i):=\{g\in\Gamma\,|\,g\equiv I_4 \bmod (1+i)\},$$

of level $(1+i)$. It is known that Γ is generated by matrices of the following three forms:

$$\begin{pmatrix}A & 0\\ 0 & (A^*)^{-1}\end{pmatrix},\ \begin{pmatrix}I_2 & B\\ 0 & I_2\end{pmatrix}\ \text{ and }\ J=\begin{pmatrix} & I_2\\ -I_2 & \end{pmatrix},$$

where $A\in GL(2,\boldsymbol{Z}[i])$ and $B=B^*\in M(2,2,\boldsymbol{Z}[i])$; $\Gamma(1+i)$ is generated by matrices of the following three forms:

$$\begin{pmatrix}A & 0\\ 0 & (A^*)^{-1}\end{pmatrix},\ \begin{pmatrix}I_2 & B\\ 0 & I_2\end{pmatrix}\ \text{ and }\ J^{-1}\begin{pmatrix}I_2 & B\\ 0 & I_2\end{pmatrix}J,$$

where $A\in GL(2,\boldsymbol{Z}[i])$, $A\equiv I_2 \bmod 1+i$, $B=B^*\in M(2,2,(1+i)\boldsymbol{Z}[i])$. We set

$$\Gamma_T:=\Gamma\rtimes\langle T\rangle,\qquad \Gamma_T(1+i):=\Gamma(1+i)\rtimes\langle T\rangle.$$

Definition. Let Λ be a subgroup of Γ_T of finite index, and $\chi:\Lambda\to\langle i\rangle := \{\pm 1, \pm i\}$ be a homomorphism. A holomorphic function f on $\boldsymbol{D}$ is called *a modular form of weight* $2k$ *relative to* Λ *with character* χ, if the following condition is satisfied:

$$f(\boldsymbol{g}\cdot W)=\chi(\boldsymbol{g})^k\{\det(CW+D)\}^{2k}f(W),$$

where $\boldsymbol{g}=gT^j\in\Lambda(j\in \boldsymbol{Z}_2)$, $g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$.

The theta function with the characteristic $(a, b)(a, b\in \boldsymbol{Z}[i]^2)$ is defined by

$$\Theta\begin{bmatrix}a\\b\end{bmatrix}(W):=\sum_{n\in \boldsymbol{Z}[i]^2}\boldsymbol{e}\left[\left(n+\frac{1}{1+i}a\right)^* W\left(n+\frac{1}{1+i}a\right)+2\operatorname{Re}\left\{\left(\frac{1}{1+i}b\right)^* n\right\}\right],$$

where $W\in\boldsymbol{D}$ and $\boldsymbol{e}[x]=\exp(\pi i x)$. These functions have the following properties:

(i) $\Theta\begin{bmatrix}\delta a\\ \varepsilon b\end{bmatrix}(W)=\Theta\begin{bmatrix}a\\b\end{bmatrix}(W)$, where $\delta, \varepsilon\in\langle i\rangle$,

(ii) $\Theta\begin{bmatrix}a+r\\ b+s\end{bmatrix}(W)=\boldsymbol{e}[\operatorname{Re}({}^t br)]\Theta\begin{bmatrix}a\\b\end{bmatrix}(W)$, where $r, s\in(1+i)\boldsymbol{Z}[i]^2$,

(iii) if ${}^t ab\notin(1+i)\boldsymbol{Z}[i]$, then $\Theta\begin{bmatrix}a\\b\end{bmatrix}(W)$ vanishes.

Proposition 1. *The theta functions are modular forms of weight* 2 *relative to* $\Gamma_T(1+i)$ *with the character* det: $\boldsymbol{g}=gT^j\to\det(g)$.

Proof. The assertion can be easily checked for each generators by the following facts.

(a) If $g\in\Gamma$ is in the form $\begin{pmatrix}A & 0\\ 0 & (A^*)^{-1}\end{pmatrix}$, $A\in GL(2, \boldsymbol{Z}[i])$, then

$$\Theta\begin{bmatrix}a\\b\end{bmatrix}(g\cdot W)=\Theta\begin{bmatrix}A^* & a\\ A^{-1} & b\end{bmatrix}(W),$$

(b) if g is in the form $\begin{pmatrix}I_2 & B\\ 0 & I_2\end{pmatrix}$, $B=B^*\in M(2,2,\boldsymbol{Z}[i])$, then

$$\Theta\begin{bmatrix}a\\b\end{bmatrix}(g\cdot W)=\boldsymbol{e}\left[\frac{1}{2}a^*Ba\right]\Theta\begin{bmatrix}a\\ \tilde{b}\end{bmatrix}(W),\quad \tilde{b}=b+Ba+(1+i)\begin{pmatrix}B_{11}\\ B_{22}\end{pmatrix},$$

(c) $$\Theta\begin{bmatrix}a\\b\end{bmatrix}(J\cdot W)=-\det(W)\Theta\begin{bmatrix}b\\a\end{bmatrix}(W),$$

(d) $$\Theta\begin{bmatrix}a\\b\end{bmatrix}(T\cdot W)=\Theta\begin{bmatrix}a\\b\end{bmatrix}(W).$$

By the properties (i), (ii) and (iii), there are only ten linearly independent theta functions; they are, for example, those with characteristics (a, b) such that $a, b\in\{0,1\}^2$ and ${}^t a\cdot b\equiv 0 \bmod 2$. The ten functions satisfy quadratic relations as we see in the following.

Proposition 2. *The theta functions satisfy the following relations*:

$$\sum_{\substack{a,b\in\{0,1\}^2\\ {}^t ab\equiv 0 \bmod 2}}\Theta\begin{bmatrix}a\\b\end{bmatrix}(W)^2\boldsymbol{e}[{}^t ca+{}^t db]=0,$$

for $c, d\in\{0,1\}^2$, ${}^t cd=1$.

Remark 3. The proposition gives six linear relations between the squares of ten linearly independent theta functions; five relations among the six are linearly independent.

To prove Proposition 2, we need the following lemma, which is a direct consequence of the orthogonality of characters.

Lemma 4. Let L_1 and L_2 be two commensurable lattices in $\boldsymbol{C}^m$. Suppose that f is a function defined on $L=L_1+L_2$ for which $\sum_{n\in L} f(n)$ is absolutely convergent. Then the summations of f over L_1 and L_2 are related as follows:

$$\sum_{n_1\in L_1} f(n_1)=[L:L_1]^{-1}\sum_{s,r}\{\sum_{n_2\in L_2} s(n_2+r)f(n_2+r)\},$$

where s run over characters of L/L_1, r run over L/L_2.

Proof of Proposition 2. Apply Lemma 4 for the data

$$L_1:=M(2,2,\boldsymbol{Z}[i]) \quad\text{and}\quad L_2:=\frac{1}{1+i}L_1\begin{pmatrix}1&1\\-1&1\end{pmatrix},$$

$$f(X):=\boldsymbol{e}\left[tr\left(\left(X+\frac{1}{1+i}M_1\right)^{*}W\left(X+\frac{1}{1+i}M_1\right)\right)+2\operatorname{Re}\left\{tr\left(\left(\frac{1}{1+i}M_2\right)^{*}X\right)\right\}\right]$$

where $X\in M(2,2,\boldsymbol{C})$, $M_1=(m_1,m_1)$, $M_2=(m_2,m_2)$ $m_j\in \boldsymbol{Z}^2$.

Let $\overline{\Gamma_T(1+i)\backslash \boldsymbol{D}}$ be the Satake compactification of the quotient space $\Gamma_T(1+i)\backslash \boldsymbol{D}$. Consider a holomorphic map $F':\Gamma_T(1+i)\backslash \boldsymbol{D}\to \boldsymbol{P}^9$ defined by

$$W\mapsto[\cdots,\Theta[j](W)^2,\cdots],$$

where the $\Theta[j]$'s are ten linearly independent theta functions. By Remark 3, the image is in a 4-dimensional linear subspace of $\boldsymbol{P}^9$, which will be denoted by Y. Let $F:L_T(1+i)\backslash \boldsymbol{D}\to Y$ be the map induced by F'. Now we state the main result of the present paper.

Theorem 5. *The map F extends to an isomorphism between $\overline{\Gamma_T(1+i)\backslash \boldsymbol{D}}$ and $Y(\simeq \boldsymbol{P}^4)$.*

Proof. (1) We first show that the map F is well defined. We have to study the zeros of the theta functions. Consider the theta function

$$\Theta[1]:=\Theta\begin{bmatrix}1\\1\\1\\1\end{bmatrix}(W)$$

and the involution

$$\gamma:=\begin{pmatrix}i&&&\\&1&&\\&&i&\\&&&1\end{pmatrix}T\in\Gamma_T(1+i).$$

The transformation formula (a) in Proposition 1 and the property (ii) leads to

$$\Theta[1]\begin{pmatrix}w_{11}&w_{12}\\w_{21}&w_{22}\end{pmatrix}=-\Theta[1]\begin{pmatrix}w_{11}&-iw_{21}\\iw_{12}&w_{22}\end{pmatrix}.$$

Hence $\Theta[1]$ has zeros along the variety $\Gamma_T(1+i)\cdot S$ where

$$S:=\{W=(W_{jk})\in \boldsymbol{D}\mid w_{21}=iw_{12}\}$$

is the set of fixed points of γ. It is proved in [2] that they are the only zeros of $\Theta[1]$ and that they are simple. Since any non-zero theta function is transformed into $\Theta[1]$, up to a non-zero factor, by an element $g\in\Gamma$, the set of zeros is a part of $\Gamma_T\cdot S$. Let us call each variety $g\cdot S$ a *mirror* for $g\gamma g^{-1}\in\Gamma_T$. One can show that, for any point in $\boldsymbol{D}$, at most four mirrors pass through the point. Thus F is well defined on $\boldsymbol{D}$ modulo $\Gamma_T(1+i)$. On the boundary of type $W=\begin{pmatrix} w & 0 \\ 0 & i\infty \end{pmatrix}$, the theta function $\Theta\begin{bmatrix} a \\ b \end{bmatrix}(W)$ reduces to $\vartheta\begin{bmatrix} a1 \\ b1 \end{bmatrix}(w)^2\ \vartheta\begin{bmatrix} a2 \\ b2 \end{bmatrix}(i\infty)^2$, where $a=\begin{pmatrix} a1 \\ a2 \end{pmatrix}$, $b=\begin{pmatrix} b1 \\ b2 \end{pmatrix}$ and ϑ is Jacobi's theta constant. The behavior of $\vartheta(w)$ shows that the map F is well defined on the boundary. Since every $\Gamma_T(1+i)$-rational boundary component can be transformed into the above by an action of Γ_T, we conclude that F is well defined on the Satake compactification.

(2) We next show that F is locally biholomorphic in $\Gamma_T(1+i)/\boldsymbol{D}$. Let $P\in\boldsymbol{D}$ be the intersection of four mirrors which are sets of zeros of four theta functions, say, $\Theta[1],\cdots,\Theta[4]$. Since each of the four theta functions $\Theta[j]$ $(1\leq j\leq 4)$ has simple zeros along the corresponding mirrors, which can be seen to be normally crossing, four functions $\Theta[j](W)^2$ $(1\leq j\leq 4)$ can be regarded as a system of local coordinates of $\Gamma_T(1+i)\backslash\boldsymbol{D}$ at the projection $\bar{P}$ of P. Thus F is locally biholomorphic at $\bar{P}$.

(3) Finally we prove that F is bioholomorphic. Since F is an open map, F is a covering map of $\boldsymbol{P}^4$. In the situation of (2), one can see that the point P is the unique intersection point of the four mirrors. Thus we have $F^{-1}(F(\bar{P}))=\{\bar{P}\}$, which implies that the sheet number of the covering map is 1 and that F is biholomorphic.

Let $\mathrm{Mod}_{2k}(1+i)$ denote the vector space of modular forms of weight $2k$ relative to $\Gamma_T(1+i)$ with character det and let $\mathrm{Mod}(1+i)$ be the graded ring:

$$\mathrm{Mod}(1+i):=\bigoplus_{k\geq 0}\mathrm{Mod}_{2k}(1+i).$$

By Theorem 5 we can easily lead the following corollary:

Corollary 6. *Any five linearly independent modular forms, which are squares of theta functions, are free generators of the graded ring* $\mathrm{Mod}(1+i)$.

Remark 7. The isomorphism $F:\overline{\Gamma_T(1+i)\backslash\boldsymbol{D}}\rightarrow Y$ $(\simeq\boldsymbol{P}^4)$ connects the analytic moduli and the algebraic moduli of a 4-dimensional family of K3 surfaces which are double covers of $\boldsymbol{P}^2$ branching along 6 lines. For more details, see [8] and [6].

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