39. *On a Certain Fractional Operator*

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The object of the present paper is to derive some properties of a certain fractional operator $J_{0,z}^{\alpha,\beta,\eta}$ defined by using the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ for analytic functions in the unit disk.

1. Introduction. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z:|z|<1\}$. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be *starlike of order* α if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{1.2}$$

for some α $(0\leqq\alpha<1)$ and for all $z\in\mathcal{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α. Further, a function $f(z)$ belonging to the class $\mathcal{A}$ is said to be *convex of order* α if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \tag{1.3}$$

for some α $(0\leqq\alpha<1)$ and for all $z\in\mathcal{U}$. Also, we denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all such functions. We note that $f(z)\in\mathcal{K}(\alpha)$ if and only if $zf'(z)\in\mathcal{S}^*(\alpha)$.

Let the functions $f_j(z)$ be defined by

$$f_j(z)=\sum_{n=0}^{\infty}a_{j,n+1}z^{n+1} \qquad (j=1,2). \tag{1.4}$$

We denote by $f_1*f_2(z)$ the Hadamard product or convolution of two functions $f_1(z)$ and $f_2(z)$, that is,

$$f_1*f_2(z)=\sum_{n=0}^{\infty}a_{1,n+1}a_{2,n+1}z^{n+1}. \tag{1.5}$$

Also, let the function $\phi(a,c;z)$ be defined by

$$\phi(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1} \qquad (z\in\mathcal{U}), \tag{1.6}$$

where $c\neq0,-1,-2,\cdots$, and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n=\begin{cases}1 & (\text{if } n=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\text{if } n\in\mathcal{N}=\{1,2,3,\cdots\}).\end{cases} \tag{1.7}$$

The function $\phi(a,c;z)$ is an incomplete beta function with

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(1.8) $$\phi(a, c; z)=z{}_2F_1(1, a; c; z).$$

Corresponding to the function $\phi(a, c; z)$, Carlson and Shaffer [1] defined a linear operator on $\mathcal{A}$ by

(1.9) $$L(a, c)f(z)=\phi(a, c; z)*f(z)$$

for $f(z)\in\mathcal{A}$. Then $L(a, c)$ maps $\mathcal{A}$ onto itself. Further, if $a\neq 0, -1, -2, \cdots$, then $L(c, a)$ is an inverse of $L(a, c)$. Also, we observe that

(1.10) $$\mathcal{K}(\alpha)=L(1, 2)\mathcal{S}^*(\alpha) \qquad (0\leqq\alpha<1)$$

and

(1.11) $$\mathcal{S}^*(\alpha)=L(2, 1)\mathcal{K}(\alpha) \qquad (0\leqq\alpha<1).$$

In order to introduce our fractional operator $J_{0,z}^{\alpha,\beta,\eta}$, we need the following definition of fractional integral operators due to Srivastava, Saigo and Owa [3].

Definition. For real numbers $\alpha>0$, β, and η, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by

(1.12) $$I_{0,z}^{\alpha,\beta,\eta}f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}{}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}\right)f(\zeta)d\zeta,$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z)=O(|z|^{\varepsilon}) \qquad (z\rightarrow 0),$$

where

$$\varepsilon>\max\{0, \beta-\eta\}-1,$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Using the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$, we introduce the *fractional operator* $J_{0,z}^{\alpha,\beta,\eta}$ defined by

(1.13) $$J_{0,z}^{\alpha,\beta,\eta}f(z)=\frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)}z^{\beta}I_{0,z}^{\alpha,\beta,\eta}f(z)$$

for $f(z)\in\mathcal{A}$. Then we observe that

(1.14) $$J_{0,z}^{\alpha,\beta,\eta}f(z)=L(2, 2-\beta)L(2-\beta+\eta, 2+\alpha+\eta)f(z).$$

2. Some properties of the fractional operator. We begin with the statement of the following lemma due to Carlson and Shaffer [1].

Lemma 1. *If* $\alpha\leqq\beta\leqq 1$ *and* $\alpha<1$, *then*

(2.1) $$L(2-2\beta, 2-2\alpha)\mathcal{S}^*(\alpha)\subset\mathcal{S}^*(\beta)\subset\mathcal{S}^*(\alpha).$$

Applying the above lemma, we derive

Theorem 1. *If* $\alpha>0$, $0\leqq\beta<1$, *and* η *is real, then*

(2.2) $$L(2+\alpha+\eta, 2-\beta+\eta)J_{0,z}^{\alpha,\beta,\eta}\mathcal{K}(1/2)\subset\mathcal{S}^*(1/2).$$

Proof. It follows from (1.10) and (1.14) that

$$\begin{aligned}J_{0,z}^{\alpha,\beta,\eta}\mathcal{K}(1/2)&=L(2, 2-\beta)L(2-\beta+\eta, 2+\alpha+\eta)\mathcal{K}(1/2)\\&=L(2, 2-\beta)L(2-\beta+\eta, 2+\alpha+\eta)L(1, 2)\mathcal{S}^*(1/2)\\&=L(1, 2-\beta)L(2-\beta+\eta, 2+\alpha+\eta)\mathcal{S}^*(1/2).\end{aligned}$$

This implies that

(2.3) $$L(2+\alpha+\eta, 2-\beta+\eta)J_{0,z}^{\alpha,\beta,\eta}\mathcal{K}(1/2)=L(1, 2-\beta)\mathcal{S}^*(1/2).$$

Noting that $\mathcal{S}^*(1/2)\subset\mathcal{S}^*(\beta/2)$ for $0\leqq\beta<1$, we have

(2.4) $$L(1, 2-\beta)\mathcal{S}^*(1/2) \subset L(1, 2-\beta)\mathcal{S}^*(\beta/2) \qquad (0 \leqq \beta < 1).$$

Therefore, with the aid of Lemma 1, we see that

(2.5) $$L(1, 2-\beta)\mathcal{S}^*(\beta/2) \subset \mathcal{S}^*(1/2) \subset \mathcal{S}^*(\beta/2)$$

which completes the proof of Theorem 1.

A function $f(z)$ in the class $\mathcal{A}$ is said to be prestarlike of order α ($\alpha \leqq 1$) if and only if

(2.6) $$\begin{cases} \dfrac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathcal{S}^*(\alpha) & (\text{for } \alpha < 1) \\ \mathrm{Re}\left(\dfrac{f(z)}{z}\right) > \dfrac{1}{2} & (\text{for } \alpha = 1). \end{cases}$$

We denote by $\mathcal{R}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions which are prestarlike of order α. The class $\mathcal{R}(\alpha)$ was introduced by Ruscheweyh [2].

In view of the definition for the class $\mathcal{R}(\alpha)$, we see that

(2.7) $$\mathcal{R}(\alpha) = L(1, 2-2\alpha)\mathcal{S}^*(\alpha) \qquad (\text{for } \alpha < 1)$$

and

(2.8) $$\mathcal{R}(1) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2},\ z \in \mathcal{U} \right\}.$$

Finally, we prove

Theorem 2. *If $\alpha > 0$, $0 \leqq \beta < 2$, and η is real, then*

(2.9) $$L(2+\alpha+\eta, 2-\beta+\eta) J_{0,z}^{\alpha,\beta,\eta} \mathcal{K}(\beta/2) = \mathcal{R}(\beta/2).$$

Proof. Noting that

$$\begin{aligned} J_{0,z}^{\alpha,\beta,\eta} \mathcal{K}(\beta/2) &= L(2, 2-\beta) L(2-\beta+\eta, 2+\alpha+\eta) \mathcal{K}(\beta/2) \\ &= L(1, 2-\beta) L(2-\beta+\eta, 2+\alpha+\eta) \mathcal{S}^*(\beta/2), \end{aligned}$$

we obtain that

(2.10) $$\begin{aligned} L(2+\alpha+\eta, 2-\beta+\eta) J_{0,z}^{\alpha,\beta,\eta} \mathcal{K}(\beta/2) &= L(1, 2-\beta) \mathcal{S}^*(\beta/2) \\ &= \mathcal{R}(\beta/2). \end{aligned}$$

Letting $\beta = 0$ in Theorem 2, we have

Corollary 1. *If $\alpha > 0$ and η is real, then*

$$L(2+\alpha+\eta, 2+\eta) J_{0,z}^{\alpha,0,\eta} \mathcal{K}(0) = \mathcal{R}(0).$$

Taking $\beta = 1$, Theorem 2 gives

Corollary 2. *If $\alpha > 0$ and η is real, then*

$$L(2+\alpha+\eta, 1+\eta) J_{0,z}^{\alpha,1,\eta} \mathcal{K}(1/2) = \mathcal{R}(1/2).$$

References

[1] B. C. Carlson and D. B. Shaffer: Starlike and prestarlike hypergeometric functions. SIAM J. Math. Anal., **15**, 737–745 (1984).

[2] St. Ruscheweyh: Linear operators between classes of prestarlike functions. Comment. Math. Helv., **52**, 497–509 (1977).

[3] H. M. Srivastava, M. Saigo and S. Owa: A class of distortion theorems involving certain operators of fractional calculus. J. Math. Anal. Appl., **131**, 412–420 (1988).