# 55. Tuboids of $C^n$ with Cone Property and Domains of Holomorphy

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**Abstract:** Let $X$ be a $C^\infty$-manifold, $M$ a closed submanifold, $\Omega$ an open set of $M$. We introduce in § 1 a class of domains $U$ of $X$ called $\Omega$-tuboids. They coincide with the original ones by [2] apart from an additional assumption, of cone type, at $\partial\Omega$. In § 2 we take a complex of sheaves $\mathcal{F}$ on $X$ and denote by $\mu_\Omega(\mathcal{F})$ the microlocalization of $\mathcal{F}$ along $\Omega$. We take a closed convex proper cone $\lambda$ of $T_M^*X$ and describe the stalk of $\boldsymbol{R}\pi_*\boldsymbol{R}\Gamma_\lambda\mu_\Omega(\mathcal{F})_{T_M^*X}$ by means of cohomology groups of $\mathcal{F}$ over $\Omega$-tuboids $U$ with profile $\gamma=\operatorname{int}\lambda^{oa}$. In § 3 we take $X=\boldsymbol{C}^n$, $M=\boldsymbol{R}^n$, $\Omega$ open convex in $M$ and prove that in the class of $\Omega$-tuboids with a prescribed profile there is a fundamental system of domains of holomorphy. By this tool we prove in § 4 a decomposition theorem for the microsupport at the boundary $SS_\Omega$ by Schapira [9] (cf. also [5]).

§ 1. Let $X$ be a $C^\infty$ manifold, $M$ a closed submanifold, let $\tau: TX\to X$ (resp $\pi: T^*X\to X$) be the tangent (resp cotangent) bundle to $X$, and let $\tau: T_MX\to M$ (resp $\pi: T_M^*X\to M$) be the normal (resp conormal) bundle to $M$ in $X$. We note that we have an embedding $\iota: TM\hookrightarrow M\times_X TX$ and a projection $\sigma: M\times_X TX\to T_MX$. For a subset $A$ of $X$ (resp of $M$) we shall define the strict normal cone of $A$ in $X$ (resp $M$) by $N^X(A)=TX\setminus C(X\setminus A, A)$ (resp $N^M(A)=TM\setminus C(M\setminus A, A)$) where $C(\cdot,\cdot)$ is the closed cone of $TX$ defined in [6]. If no confusion may arise, we shall omit the superscripts $X$ and $M$. Let $\Omega$ be an open set of $M$ and $x_0$ a point of $\partial\Omega$. We shall assume

$$N_{x_0}^M(\Omega)\neq\emptyset. \tag{1.1}$$

Let $\gamma$ be an open convex cone of $\bar\Omega\times_M T_MX$ with $\tau(\gamma)\supset\bar\Omega$.

**Definition 1.1.** A domain $U\subset X$ is said to be an $\Omega$-tuboid with profile $\gamma$ when

$$\sigma(M\times_X TX\setminus C(X\setminus U, \bar\Omega))\supset\gamma. \tag{1.2}$$

One proves that $\theta\in T_{x_0}X\setminus C_{x_0}(X\setminus U,\bar\Omega)$ iff for a choice of local coordinates there exists a neighborhood $V$ of $x_0$ and an open cone $G$ containing $\theta$ s.t. $((\bar\Omega\cap V)+G)\cap V\subset U$. In particular:

$$TX\setminus C(X\setminus U,\bar\Omega)=(TX\setminus C(X\setminus U,\bar\Omega))+N(\Omega).$$

**Lemma 1.2.** *Let* (1.2) *hold. Then there exists an open convex cone* $\beta\subset\bar\Omega\times_X TX$:

$$\beta\subset TX\setminus C(X\setminus U,\bar\Omega),\quad \beta=\beta+N(\Omega),\quad \sigma(\beta)\supset\gamma. \tag{1.3}$$

*Proof.* For a choice of coordinates on $X$ we identify

$$M\times_X TX\cong TM\oplus_M T_MX\ni(t, x+\sqrt{-1}\,y). \tag{1.4}$$

Let $\theta \in N_t(\Omega)$, $|\theta|=1$ and let $\gamma' \subset\subset \gamma$ (in the sense that $\overline{\gamma'/\boldsymbol{R}^+} \subset\subset \gamma$). Owing to (1.2) we then have for suitable $\varepsilon$

(1.5) $$T_tX \setminus C_t(X \setminus U, \bar{\Omega}) \supset \boldsymbol{R}^+(\theta + (\gamma'_t)_\varepsilon) \ \forall t,$$

(where $(\gamma'_t)_\varepsilon = \{y \in \gamma'_t : |y| < \varepsilon\}$). One may find an open cone $\beta \subset TX$ with convex fibers such that $\forall t$:

$$\beta_t \subset T_tX \setminus C_t(X \setminus U, \bar{\Omega}), \quad C_\theta(\beta_t, \{\theta\}) \supset \gamma_t.$$

In particular $\sigma(B_t) \supset \gamma_t$. If we replace $\beta$ by $\beta + N(\Omega)$ we get the conclusion.

Let $\gamma \subset T_MX$, $\beta \subset M \times_X TX$, $\alpha \subset TM$ be open (convex) cones with $\beta = \beta + \alpha$. Then

**Lemma 1.3.** *We have*

(1.6) $$\sigma(\beta) \supset \gamma \Leftrightarrow \forall \gamma' \subset\subset \gamma \ \exists \beta' \ \text{open convex}: \ \beta' \supset \alpha \ \text{and} \ \beta \supset \beta' \cap \sigma^{-1}(\gamma').$$

*Proof.* In the coordinates of (1.4) and for $\theta \in N_t(\Omega)$, $|\theta|=1$, we have:

(1.7) $$\beta_t \supset \boldsymbol{R}^+(\theta + (\gamma'_t)_\varepsilon) + \alpha_t \supset \sigma^{-1}(\gamma'_t) \cap \{x + \sqrt{-1}y \, ; \, x \in \alpha_t, |y| < \varepsilon' \operatorname{dist}(x, \partial\alpha_t)\} = \sigma^{-1}(\gamma'_t) \cap \beta'_t$$

**Proposition 1.4.** *Condition* (1.2) *is equivalent, for a choice of coordinates* $x + \sqrt{-1}y \in X \cong T_MX$ *to*:

(1.8) $$U \supset \{x + \sqrt{-1}y \in \Omega \times_M \gamma' : |y| < \varepsilon\delta_x\} \ \forall \gamma' \subset\subset \gamma \quad \text{and for suitable } \varepsilon$$

(*where* $\delta_x = \operatorname{dist}(x, \partial\Omega) \wedge 1$).

*Proof.* The proof just consists in rephrasing Lemma 1.3 with $\alpha = N(\Omega)$.

§ 2. Let $X$ be a $C^\infty$-manifold of dimension $n$, $M$ a closed submanifold of $X$ of codimension $l$, and let $TM \xrightarrow{\iota} M \times_X TX \xrightarrow{\sigma} T_MX$ and $T^*M \xleftarrow{\rho} M \times_X T^*X \xrightarrow{\varpi} T^*X$ be the natural mappings. We shall consider the families of open convex cones $\gamma \subset T_MX$ (or $\alpha \subset TM$ or $\beta \subset TX$) and closed convex proper cones $\lambda \subset T^*_MX$ (or $\nu \subset T^*M$ or $\mu \subset T^*X$). They are related by $\lambda = \gamma^\circ$ (or $\nu = \alpha^\circ$, $\mu = \beta^\circ$), where $\gamma^\circ$ ($\alpha^\circ$, $\beta^\circ$) denote the polar cone to $\gamma$ ($\alpha$, $\beta$). It is immediate to prove that:

(2.1) $$\sigma(\beta) \supset \gamma \Leftrightarrow \mu \cap T^*_MX \subset \lambda$$
$$\bar{\beta} \supset \alpha \Leftrightarrow \mu \subset \rho^{-1}(\nu).$$

One also sees that if $\rho(\beta)$ is proper, then

(2.2) $$\mu \text{ proper} \Leftrightarrow \mu \cap T^*_MX \text{ proper}$$
$$\text{c.h.}(\mu) \cap T^*_MX = \text{c.h.}(\mu \cap T^*_MX),$$

where "c.h." denotes the convex hull. We denote by $D^b(X)$ the derived category of the category of complexes of sheaves with bounded cohomology. For $\mathcal{F} \in \operatorname{Ob} D^b(X)$ and for $\Omega \subset M$ open, we put $\mu_\Omega(\mathcal{F}) = \mu\hom(\boldsymbol{Z}_\Omega, \mathcal{F})$ (where $\mu\hom(\cdot, \cdot)$ is the bifunctor of [6, 7]) and call it the microlocalization of $\mathcal{F}$ along $\Omega$. Let $x_0 \in \partial\Omega$.

**Theorem 2.1.** *Assume that* $N^M_{x_0}(\Omega) \neq \emptyset$, *let* $\lambda$ *be a closed convex proper cone of* $T^*_MX$ *containing* $\bar{\Omega} \times_X T^*_XX$ *at* $x_0$. *Then*

(2.3) $$\mathcal{H}^j_\lambda(\mu_\Omega(\mathcal{F})_{T^*_MX})_{x_0} = \varinjlim_{U,B} H^{j-l}(U \cap B, \mathcal{F})$$

*where* $U$ (*resp* $B$) *ranges through the family of tuboids with profile* $\gamma = \operatorname{int} \lambda^{\circ a}$ (*resp open neighborhoods of* $x_0$).

*Proof* (cf. also [11]). Let us denote by $\mu$ the cones of $T^*X$ with $\rho(\mu)\subset N^o(\Omega)$ and $\mu\cap T^*_MX\subset\lambda$; by (2.2) it is not restrictive to assume the $\mu$'s to be proper and convex. Let $q_j: X\times X\to X$, $j=1, 2$ be the projections, let $s: X\times X\to X$, $(x, y)\mapsto x-y$ and let $\Delta$ be the diagonal of $X\times X$. We have:

$$(2.4)\qquad H^j_\lambda(T^*_MX, \mu_\Omega(\mathcal{F})_{T^*_MX})=\varinjlim_{\mu} H^j_\mu(T^*X, \mu_\Omega(\mathcal{F}))$$
$$=\varinjlim_{W} H^{j-n}\boldsymbol{R}\mathcal{H}om_{\boldsymbol{Z}_X}(\boldsymbol{R}_{q_{1!}}\boldsymbol{Z}_{W\cap(X\times\Omega)}, \mathcal{F}),$$

for $W$ verifying $T_\Delta(X\times X)\setminus C_\Delta((X\times X)\setminus W)\supset\text{int}\,\mu^{oa}$, in the identification $T_\Delta(X\times X)\underset{s'}{\rightrightarrows} TX$. (cf. 6, Proposition 2.3.2] as for the latter equality.) But for a fundamental system of neighborhoods $B$ of $x_0$, we have:

$$(2.5)\qquad \boldsymbol{R}_{q_{1!}}\boldsymbol{Z}_{W\cap(X\times\Omega)}|_B=\boldsymbol{Z}_{q_1(W\cap(X\times\Omega))}[-\dim M]|_B.$$

If we assume (2.5) the conclusion is immediate since the sets $q_1(W\cap(X\times\Omega))$ are a fundamental system of $\Omega$-tuboids with profile int $\lambda^{oa}$ (cf. [11, Lemma 1.2 and 1.3]). Let us prove (2.5). We identify $T_xX\cong X$ and $M\times_X TX\cong T_MX\oplus TM$ (for a choice of a projection $X\to M$). Let $F\subset\subset\mu^a_x$, $N\subset\subset N^M_x(\Omega)$ $\forall x$ close to $x_0$ and put $G=F+N$, $G_\varepsilon=G\cap\{g\in G: \langle g, \theta\rangle<\varepsilon\}$ where $\theta$ is a fixed vector of $N^M_{x_0}(\Omega)$. We shall prove (2.5) with $W$ replaced by $s'^{-1}(G_\varepsilon)$. In fact set $A_x=q_1^{-1}(x)\cap s'^{-1}(G_\varepsilon)\cap(X\times\Omega)$. Let $L_\varepsilon$ be the plane through $x_0+\varepsilon\theta$ with conormal $\theta$ and $L^-_\varepsilon$ the half-space with exterior conormal $\theta$ and boundary $L_\varepsilon$. Then for suitable $B$ and $\forall x\in B$, we see that $A_x$ is an open connected set which verifies:

$$\begin{cases}(y+N)\cap L^-_{\varepsilon'}\subset A_x & \forall y\in A_x\\ (y+N)\cap(z+N)\cap L_{\varepsilon'}\neq\emptyset & \forall y, z\in A_x,\end{cases}$$

(for a new $\varepsilon'$). Hence $A_x$ is contractile and $\boldsymbol{R}\Gamma_c(A_x, \boldsymbol{Z}_M)=\boldsymbol{Z}[\dim M]$.

**Remark 2.2.** If $\Omega$ is convex in $M\cong\boldsymbol{R}^n$, then we get a "global" version of Theorem 2.1: $H^j_\lambda(T^*_MX, \mu_\Omega(\mathcal{F})_{T^*_MX})=\varinjlim_U H^{j-l}(U, \mathcal{F})$.

**§3.** We shall extend here the results of [2]. Let $\boldsymbol{C}^n=\boldsymbol{R}^n_x+\sqrt{-1}\,\boldsymbol{R}^n_y$, let $\pi=\pi_x$ be the first projection $\boldsymbol{C}^n\to\boldsymbol{R}^n_x$, let $\dot{\boldsymbol{R}}^n=\boldsymbol{R}^n\setminus\{0\}$, and set $S^{n-1}=\dot{\boldsymbol{R}}^n/\boldsymbol{R}^+$. We shall call (convex) cone of $\boldsymbol{C}^n$ any subset $\gamma$ of $\boldsymbol{C}^n$ with (convex) conic $\pi$-fibers. For cones $\gamma$, $\gamma'$ of $\boldsymbol{C}^n$, we write $\gamma'\subset\subset\gamma$ when $\bar{\gamma}\cap(\boldsymbol{R}^n_x+\sqrt{-1}S^{n-1}_y)$ is compact in $\gamma$. Let $\Omega$ be an open set of $\boldsymbol{R}^n$, and $\gamma$ an open convex cone of $\bar{\Omega}\times_M T_MX$. We shall assume all through this section that $\Omega$ is convex. For $x\in\Omega$, we set $\delta_x\overset{\text{def.}}{=}\text{dist}(x, \partial\Omega)\wedge 1$ and $\gamma_\Omega\overset{\text{def.}}{=}\gamma\cap\pi^{-1}(\Omega)$. We recall from §1 that a domain $U$ is an $\Omega$-tuboid with profile $\gamma$, when $\forall\gamma'\subset\subset\gamma\ \exists\varepsilon$: $U\supset\{x+\sqrt{-1}\,y\in\gamma'_\Omega, |y|<\varepsilon\delta_x\}$. We shall also assume without loss of generality that $U\subset\gamma_\Omega$ in what follows. Note that if $\pi(\gamma)\subset\Omega$, then our definition coincides with the original one by [2].

**Lemma 3.1.** *Let $U'\subset U$ be $\Omega$-tuboids with profiles $\gamma'\subset\subset\gamma$, and set $W'=\pi(\gamma')$, $W=\pi(\gamma)$. Assume that $U'$ has convex fibers, that $\bar{U}'\setminus\bar{\Omega}\subset U$, and that*

(3.1) *For a finite open covering $\bigcup_j V'^j\supset\partial\Omega\cap W'$, for open truncated cones $G'^j$ and $H'^j$ with $G'^j\subset\subset H'^j$, we have*

$$U'_{V'^j} \subset \bigcup_{x\in\Omega\cap V'^j} x+\sqrt{-1}G'_{\delta_x} \subset \bigcup_{x\in\Omega\cap V'^j} x+\sqrt{-1}H'_{\delta_x} \subset U.$$

*Then for any $\gamma''$ with $\gamma'\subset\subset\gamma''\subset\subset\gamma$ there exists an $\Omega$-tuboid $U''$ with profile $\gamma''$ such that: $U''$ has convex fibers; $U'\subset U''$ and $\overline{U}''\setminus\overline{\Omega}\subset U$; (3.1) holds for new $V''^j$, $G''^j$, $H''^j$.*

*Proof.* Set

$$U''=\bigcup_{x\in\Omega\cup W''} x+\sqrt{-1}\,\mathrm{c.h.}\,(U'_x\cup(\gamma''_x)_{\kappa\delta_x}).$$

According to [2], $U''$ satisfies all requirements except over a neighborhood of $\partial\Omega$. We decompose such a neighborhood as $\bigcup_j V^j$ with $V=V^j$ satisfying $V\subset\subset W$, $\gamma''_V\subset V+\sqrt{-1}F''\subset\subset\gamma$. (Observe that here the $F$'s (resp. $G$'s) are cones (resp truncated cones).) We assume also (3.1) to be satisfied by the $V$'s such that $V\cap\overline{W}'\neq\emptyset$ (and neglect the other $V$'s). We have

$$U''_V\subset\bigcup_{V\cap\Omega} x+\sqrt{-1}\,c.h.(G'_{\delta_x}\cup F''_{\delta_x}).$$

Since $\bigcap_\kappa c.h.(G'\cup F'')=\overline{G}'$, then $\forall\kappa_1$ and for suitable $\kappa$ and $J'$ with $G'\subset\subset J'\subset\subset H'$, we have that $\overline{c.h.(G'\cup F''_\kappa)}\subset J'\cup F''_{\kappa_1}$. Let $F\supset\supset F''$ with $V+\sqrt{-1}F\subset\gamma$; since $U$ has profile $\gamma$, then $U\supset\bigcup_{\Omega\cap V} x+\sqrt{-1}F_{\kappa_2\delta_x}$. Thus if we take $\kappa_1<\kappa_2$ and set $G''=J'\cup F''_{\kappa_1}$, $H''=H'\cup F_{\kappa_2}$, we get

$$\overline{U''_V}\setminus\overline{V}\subset\bigcup_{\Omega\cap V} x+\sqrt{-1}G''_{\delta_x}\subset\bigcup_{\Omega\cap V} x+\sqrt{-1}H''_{\delta_x}\subset U.$$

Reasoning by induction one immediately obtains from Lemma 3.1.

**Proposition 3.2.** *Any $\Omega$-tuboid with profile $\gamma$ contains an $\Omega$-tuboid with the same profile $\gamma$ and with convex fibers.*

Let $G$ be an open convex set of $\boldsymbol{R}^n$ contained in $\{y:|y|<1/2\}$ and with $0\in\overline{G}$. Let $S^{n-1}=\{\eta\in\boldsymbol{R}^n:|\eta|=1\}$ and define $\sigma_{\eta G}=\sup_{g\in G}\langle g,-\eta\rangle$. We also write $\sigma_\eta=\sigma_{\eta G}$ and define $\widehat{G}=\bigcap_{\eta\in S^{n-1}}\{y:\langle y,\eta\rangle+\sigma_\eta-|y+\sigma_\eta\eta|^2>0\}$ (cf. [2]). Clearly

$$\widehat{G}\subset G,\quad\text{and}\quad C(\widehat{G},\{0\})=C(G,\{0\}). \tag{3.2}$$

Let $\Omega$ be an open convex set of $\boldsymbol{R}^n_x$ and $\gamma$ an open convex cone of $\overline{\Omega}+\sqrt{-1}\boldsymbol{R}^n_y$.

**Theorem 3.3.** *Let $U$ be an $\Omega$-tuboid with profile $\gamma$. Then $U$ contains an $\Omega$-tuboid with the same profile $\gamma$ which is in addition a domain of holomorphy.*

*Proof.* It is not restrictive to assume that $U$ has convex fibers and that $U\subset\{x+\sqrt{-1}y: x\in\Omega,\ |y|<\varepsilon\delta_x\}$, $\varepsilon$ small. Let $\Omega$ be defined by $\phi(x)<0$ for $-\phi$ being a convex function; clearly $-\phi(x)$ is equivalent to $\delta_x$ over $K\cap\Omega$ $(K\subset\subset\boldsymbol{R}^n)$. We also remark that $\boldsymbol{C}^n\to\boldsymbol{R}$, $z\mapsto-\phi\,(\mathrm{Re}\,z)$ is plurisubharmonic. For $a\in\Omega\cap\pi(\gamma)$ and $\eta\in S^{n-1}$, we write $\sigma_{\eta a}=\sigma_{\eta U_a}$ $(=\sup_{y\in U_a}\langle y,-\eta\rangle)$, and let $\psi_{a\eta}(x,y)\overset{\text{def.}}{=}\langle y,\eta\rangle+\sigma_{a\eta}(\phi(x)/\phi(a))-|y+\sigma_{a\eta}\eta|^2+|x-a|^2$. We define

(3.3)

$$U'=\{x+\sqrt{-1}y: x\in\Omega\cap\pi(\gamma)\text{ and }\psi_{a\eta}(x,y)>0\ \forall a\in\Omega\cap\pi(\gamma),\text{ and }\forall\eta\in S^{n-1}\}.$$

Clearly $U'_x\subset\widehat{U}_x\subset U_x\,\forall x$. Moreover:

$$\{x+\sqrt{-1}y:\psi_{a\eta}(x,y)>0\}\supset\left\{x+\sqrt{-1}y:|y|<\left(|x-a|^2+\left(\sigma_{a\eta}\frac{\phi(x)}{\phi(a)}-\sigma^2_{a\eta}\right)+\frac{1}{4}\right)^{1/2}-\frac{1}{2}\right\}. \tag{3.4}$$

By (3.4) one proves as in [2] that $U'$ is an open domain. It remains to prove that it is an $\Omega$-tuboid of profile $\gamma$. Let $W'+G'\subset\subset\gamma$. Fix $x_0\in W''\subset\subset W'$, fix $\varepsilon>0$ and define $U_1'$ (resp $U_2'$, resp $U_3'$) by adding in the definition (3.3) for $U'$ the condition $a\in W'$ and $(\phi(x_0)/\phi(a)^2)\geq\varepsilon$, (resp $a\in W'$ and $(\phi(x_0)/\phi(a))<\varepsilon$, resp $a\notin W'$); hence $U'=U_1'\cup U_2'\cup U_3'$. By the second of (3.2) one gets $(U_1')_{x_0}\supset G'_{\kappa\delta_{x_0}}$ for suitable $\kappa$ independent of $x_0\in W''$. One also easily sees that $(U_2')_{x_0}\supset S^{n-1}_{\kappa\delta_{x_0}}$, and $(U_3')_{x_0}\supset S^{n-1}_{\kappa d^2}$ for $d=\operatorname{dist}(\partial W',\partial W'')$. The conclusion follows.

**§4.** Let $M$ be a $C^\omega$-manifold of dimension $n$, $X$ a complexification of $M$, $\Omega$ an open $C^\omega$-convex subset of $M$. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X$, and $\mathrm{or}_{M/X}$ the sheaf of relative orientation of $M$ in $X$. We shall deal with the complex by Schapira (see also [5]) of microfunctions at the boundary

$$\mathcal{C}_{\Omega|X}=\mu\hom(\mathbf{Z}_\Omega,\mathcal{O}_X)\otimes \mathrm{or}_{M/X}[n]. \tag{4.1}$$

Let $x\in\partial\Omega$, let $\lambda$ be a closed convex proper cone of $\bar{\Omega}\times_M T^*_MX$ such that $\pi(\lambda)$ is a neighborhood of $x$ in $\bar{\Omega}$ and set $\gamma=\operatorname{int}\lambda^{oa}$. The results of §2 and 3 give

**Theorem 4.1.** *We have*

$$\mathcal{H}^i_\lambda((\mathcal{C}_{\Omega|X})_{T^*_MX})_x=\begin{cases}0 & \text{for } i\neq 0\\ \varinjlim_U \Gamma(U,\mathcal{O}_X) & \text{for } i=0,\end{cases} \tag{4.2}$$

*where $U$ ranges through the family of $\Omega$-tuboids of holomorphy of $X$ with profile $\gamma$.*

We assume now that

$$(\mathcal{C}_{\Omega|X})_{T^*_MX} \text{ is concentrated in degree } 0 \tag{4.3}$$

(cf. [9] and [3] for sufficient condition for (4.3) to hold). Let $\mathcal{B}_M$ be the sheaf of hyperfunctions on $M$, let $\iota:\Omega\hookrightarrow M$ be the embedding, and let $\Gamma_\Omega(\mathcal{B}_M)\overset{\text{def.}}{=}\iota_*\iota^{-1}(\mathcal{B}_M)$. We recall that $\pi_*((\mathcal{C}_{\Omega|X})_{T^*_MX})=\Gamma_\Omega(\mathcal{B}_M)$. We also recall that for $f\in\Gamma_\Omega(\mathcal{B}_M)$ the microsupport at the boundary $SS_\Omega(f)$ is the support of $f$ identified to a section of $(\mathcal{C}_{\Omega|X})_{T^*_MX}$. We then get

**Proposition 4.2.** *Let $f\in\Gamma_\Omega(\mathcal{B}_M)_x$ and let $\lambda_j$, $j=1,\cdots,s$ be a family of closed convex proper cones with $\bigcup_{j=1}^s\lambda_j\supset SS_\Omega(f)$ and with $\pi(\lambda_j)$ being a neighborhood of $x$ in $\bar{\Omega}$ $\forall j$. Then we may find $f_j\in\Gamma_\Omega(\mathcal{B}_M)_x$, $j=1,\cdots,s$ such that $f=\sum_{j=1}^s f_j$ and $SS_\Omega(f_j)\subset\lambda_j$.*

*Proof.* One sees that the property: $\mathcal{H}^i_\lambda((\mathcal{C}_{\Omega|X})_{T^*_MX})_x=0$ $\forall i\geq 1$, proved in Theorem 4.1, is stable under finite intersection and finite union of $\lambda$'s. (The first is trivial while the second is an easy application of the Mayer-Vietoris long exact sequence.) The conclusion follows at once.

Some decomposition theorem of the above type was already stated, in a different frame in [8].

**Corollary 4.3.** *Let $f\in\Gamma_\Omega(\mathcal{B}_M)_x$ and let $p\in\dot{T}^*_MX$, $\pi(p)=x$. Then $p\notin SS_\Omega(f)$ if and only if $f$ is a finite sum of boundary values of holomorphic functions $F_j\in\mathcal{O}_X(U_j)$ with the $U_j'$s being $\Omega$-tuboids whose profiles $\gamma_j$ verify $\gamma_j^{oa}\not\ni p$.*

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