86. A Note on Class Groups of Abelian Number Fields

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Let A be any finite abelian group. It is known that there exist (many) abelian number fields whose class groups contain a subgroup isomorphic to A. Known such fields K are all imaginary and have large Galois group $Gal(K/\boldsymbol{Q})$ (see [1], [5], Cor. 3.9)*). In this note we show that there exist infinitely many real abelian number fields having such a property whose Galois group $Gal(K/\boldsymbol{Q})$ are elementary abelian:

Theorem 1. *For any finite abelian group A, there exist infinitely many real elmentary 2-abelian number fields K (i.e. real abelian extensions of $\boldsymbol{Q}$ whose Galois groups are of type $(2,2,\cdots,2)$) whose class groups contain a subgroup isomorphic to A. Moreover, among these fields we can take a sequence $K_1, K_2, \cdots$ such that $(d_{K_i}, d_{K_j})=1$ if $i\neq j$ and all of the 2-rank of $Gal(K_i/\boldsymbol{Q})$ are equal to rank A. Here, d_{K_i} denotes the discriminant of K_i.*

We can prove this easily using genus theory and the following theorem on the divisibility of class numbers of real quadratic number fields:

Theorem Y (Yamamoto [5], Part I, Th. 2). *For any natural number n and disjoint finite sets of prime numbers S_1, S_2, S_3, there exist infinitely many real quadratic number fields F such that*

(a) *The class group of F contains a subgroup isomorphic to $\boldsymbol{Z}/n\boldsymbol{Z}$.*

(b) *All primes contained in S_i* $\begin{cases} \textit{are decomposed in } F & (i=1), \\ \textit{remain prime in } F & (i=2), \\ \textit{are ramified in } F & (i=3). \end{cases}$

Proof of Theorem 1. By the structure theorem for finite abelian groups, we may write

$$A\cong \boldsymbol{Z}/n_1\boldsymbol{Z}\times\boldsymbol{Z}/n_2\boldsymbol{Z}\times\cdots\times\boldsymbol{Z}/n_r\boldsymbol{Z},\quad n_1|n_2|\cdots|n_r,$$

for some natural numbers $n_1, n_2, \cdots, n_r$. For each $i=1,2,\cdots,r$, we take a real quadratic number field F_i whose class group contains a subgroup isomorphic to $\boldsymbol{Z}/n_i\boldsymbol{Z}$. We can take these fields such that $(d_{F_i}, d_{F_j})=1$ if $i\neq j$ by the condition (b) of Theorem Y. Put $K=F_1F_2\cdots F_r$ (the composite field) and then K is a real elementary 2-abelian field satisfying the required property. Let L_i be an unramified abelian extension of F_i such that $Gal(L_i/F_i)\cong\boldsymbol{Z}/n_i\boldsymbol{Z}$. By genus theory the maximal extension of F_i which is unramified at all finite primes and abelian over $\boldsymbol{Q}$ is obtained by

*) Recently, Osada[3] proved the existence of the maximal real subfield of a cyclotomic field with such a property. But, this field has also large Galois group and his result is obtained from ours as a corollary.

composing F_i with all quadratic number fields whose discriminants are prime discriminant divisors of d_{F_i}. Therefore, from the definition of K, L_i and K are linearly independent over F_i. Hence L_iK/K is an unramified abelian extension such that $Gal(L_iK/K)\cong \boldsymbol{Z}/n_i\boldsymbol{Z}$. Since $(d_{F_i}, d_{F_j})=1$, if $i\neq j$, L_iK and L_jK are linearly independent over K. Therefore, if we put $M=L_1L_2\cdots L_rK$, then M/K is an unramified abelian extension such that $Gal(M/K)\cong A$. The infiniteness is proved by using the condition (b) of Theorem Y.

For any odd prime number l, if we have a corresponding result to Theorem Y on cyclic number fields of degree l, we can generalize Theorem 1 as follows:

"Let l be any prime number. For any abelian group A, there exist infinitely many (real) elementary l-abelian number fields whose class groups contain a subgroup isomorphic to A. Moreover, among these fields we can take a sequence $K_1, K_2, \cdots$ such that $(d_{K_i}, d_{K_j})=1$ if $i\neq j$ and all of the l-rank of $Gal(K_i/\boldsymbol{Q})$ are equal to rank A."

For $l=3$, Uchida [4] proved a corresponding result, and Nakano improved his result:

Theorem N (Nakano [2]). *For any natural number n, there exist infinitely many cyclic cubic number fields whose class groups contain a subgroup isomorphic to $(\boldsymbol{Z}/n\boldsymbol{Z})^2$.*

From Nakano's proof we can impose on the fields conditions on ramification. Therefore, we have

Theorem 2. *For any finite abelian group A, there exist infinitely many elementary 3-abelian number fields whose class groups contain a subgroup isomorphic to A. Moreover, among these fields we can take a sequence $K_1, K_2, \cdots$ such that $(d_{K_i}, d_{K_j})=1$ if $i\neq j$ and all of the 3-rank of $Gal(K_i/\boldsymbol{Q})$ are at most* [rank A/2].

References

[1] G. Cornell: Abhyankar's lemma and the class group. Number Theory, Carbondale 1979 (Proc. Southern Illinois Conf., Southern Illinois Univ., Carbondale, Ill., 1979). Lect. Notes in Math., vol. 751, Springer, Berlin, pp. 82–88 (1979).

[2] S. Nakano: Ideal class groups of cubic cyclic fields. Acta Arith., **46**, 297–300 (1986).

[3] H. Osada: Note on class number of real cyclotomic number field. III (preprint).

[4] K. Uchida: Class numbers of cubic cyclic fields. J. Math. Soc. Japan, **26**, 447–453 (1974).

[5] L. C. Washington: Introduction to Cyclotomic Fields. Graduate Text in Mathematics. vol. 83, Springer-Verlag, New York, Berlin (1980).

[6] Y. Yamamoto: On unramified Galois extensions of quadratic number fields. Osaka J. Math., **7**, 57–76 (1970).