26. *Some Observations Concerning the Distribution of the Zeros of the Zeta Functions. III*

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(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1992)

§ 1. **Introduction.** The purpose of the present article is to give a refinement on our previous results (e.g. Theorems 6 and 7 in pp. 141–142 of [4] and a special case of Theorem in p. 100 of [1]) on the exponential sums over the zeros of some zeta functions. Some of the consequences resulting from this will also be mentioned.

Let $L(s, \psi)$ be a Dirichlet L-function with a primitive Dirichlet character ψ mod $k \geq 1$. When $k=1$, we suppose that $L(s, \psi)=\zeta(s)=$ the Riemann zeta function. Let $\gamma(\psi)$ run over the imaginary parts of the zeros of $L(s, \psi)$. We shall prove the following theorem under the Generalized Riemann Hypothesis (G.R.H.) for $L(s, \psi)$.

Theorem. (Under G.R.H.) *Suppose that* $0<\alpha \ll T$, $1 \ll Y \leq T$ *and* b *be any positive number. Then we have*

$$\sum_{Y<\gamma(\psi)\leq T} e^{ib\gamma(\psi)\log(b\gamma(\psi)/2\pi e\alpha)} = -e^{(\pi/4)i}\frac{\sqrt{\alpha}}{b}\sum_{(Yb/2\pi\alpha)^b\leq n\leq (Tb/2\pi\alpha)^b}\frac{\Lambda(n)\psi(n)}{n^{1/2-1/2b}}e^{-2\pi i\alpha n^{1/b}}$$
$$+O\left(\mathrm{Min}\left(\frac{1}{|\log Y/\alpha|}, \sqrt{\alpha}+1\right)\log kT\right)+O\left(\frac{\log kT}{\log\log kT}\right)$$
$$+O(B(b,k,\alpha,T))+O\left(C(b,\alpha,Y,T)\log\frac{T}{\alpha}\right),$$

where $\Lambda(n)$ *is the von-Mangoldt function,*

$$C(b,\alpha,Y,T)=\begin{cases}\left(\dfrac{\alpha}{Y}\right)^{b/2}\sqrt{Y} & \text{if } b>1\\ \sqrt{\alpha}\log\dfrac{T}{Y} & \text{if } b=1\\ \eta_\alpha(Y)\sqrt{T}\left(\dfrac{\alpha}{T}\right)^{b/2}+(1-\eta_\alpha(Y))\sqrt{\alpha} & \text{if } 0<b<1,\end{cases}$$

$\eta_\alpha(Y)=1$ *if* $\alpha<CY$, *and* $=0$ *if* $\alpha \geq CY$ *with some positive constant* C *and* $B(b,k,\alpha,T)$

$$=\mathrm{Min}\left(\left(\frac{T}{\alpha}\right)^{b/2}\log\frac{T}{\alpha}\frac{\log kT}{(\log\log kT)^2}, \left(\frac{T}{\alpha}\right)^{b/(\log\log kT)}\frac{\log kT}{\log((\log kT)/(T/\alpha)^b+2)}\right)$$
$$+\left(\frac{T}{\alpha}\right)^{b/2}\left(\log\frac{T}{\alpha}\log\log\frac{T}{\alpha}+\sqrt{\frac{\log kT}{\log\log kT}}\frac{1}{\log((T/\alpha)^b/(\log kT\cdot\log\log kT)+2)}\right.$$
$$\left.+\frac{\log(T/\alpha)}{T}\frac{\log kT}{(\log\log kT)^2}\right).$$

If we ignore the dependence on k and α, we get the following simpler expression.

Cor. 1. (Under G.R.H.) *For $T>T_0$ and for any positive b and α, we have*

$$\sum_{0<\gamma(\psi)\le T} e^{ib\gamma(\psi)\log(b\gamma(\psi)/2\pi e\alpha)} = -e^{(\pi/4)i}\frac{\sqrt{\alpha}}{b}\sum_{2\le n\le(Tb/2\pi\alpha)^b}\frac{\Lambda(n)\psi(n)}{n^{1/2-1/2b}}e^{-2\pi i\alpha n^{1/b}}$$
$$+O(T^{b/2}\log T\cdot\log\log T)+O(T^{1/2-b/2}\log T).$$

These improve upon our previous results in [1] and [4] as stated above and should be also compared with Schoissengeier's results in [9]. It enables us to refine all of our results which have been proved using our previous estimate on the above sum. We may pick up the following corollaries in the case of $\zeta(s)$. We denote the imaginary parts of zeros $\zeta(s)$ by γ.

Cor. 2. (Under the Riemann Hypothesis) *Let b and α be positive numbers. Suppose that positive θ satisfies*

$$\sum_{n<y}\Lambda(n)e^{2\pi i\alpha n^{1/b}}\ll y^{\theta}\qquad \textit{for } y>y_0.$$

Then, we have

$$\sum_{0<\gamma\le T} e^{ib\gamma\log(b\gamma/2\pi e\alpha)}\ll T^{b(\theta-1/2)+1/2}+T^{b/2}\log T\cdot\log\log T.$$

In particular, we see, using Piateski-Shapiro's constant $\theta=11/12+\varepsilon$ (cf. [8]),

$$\sum_{0<\gamma\le T} e^{ib\gamma\log(b\gamma/2\pi e\alpha)}=o(T\log T)$$

for $0<b<6/5$. We notice next the following corollary.

Cor. 3. (Under the Riemann Hypothesis) *For any integer $K\ge 1$, G.R.H. for all Dirichlet L-function $L(s,\chi^K)$ with a Dirichlet character χ* mod $q\ge 3$ *is equivalent to the relation*

$$\sum_{0<\gamma\le T} e^{i(\gamma/K)\log(\gamma/(2\pi eK(a/q)))} = -e^{(\pi/4)i}C\left(\frac{a}{q},K\right)\left(\frac{T}{2\pi}\right)^{(1/2)(1+1/K)}+O(T^{1/2+\varepsilon})$$

for any positive ε and any integer a satisfying $1\le a\le q$ and $(a,q)=1$ and $T>T_0$, where we put

$$C\left(\frac{a}{q},K\right)=\frac{2K^{(1/2)(1-1/K)}}{(K+1)\varphi(q)}\left(\frac{a}{q}\right)^{-1/2K}\sum_{b=1,(b,q)=1}^{q}e^{-2\pi i(a/q)b^K}$$

with the Euler function $\varphi(q)$.

We should notice that we have proved this for $K\ge 5$ in [2]. Thus we have fixed our problem left open for the case $K=1,2,3$ and 4 (cf. p. 130 of [4], for example).

Some improvements on the remainder terms in the asymptotic formulas for the sums like

$$\sum_{0\le\gamma\le T}\zeta'\left(\frac{1}{2}+i\gamma\right)$$

can be also obtained, although we shall not describe them here.

§ 2. **Proof of Theorem.** We follow the argument in pp. 143–155 of [4]. We put $f(y)=by\log(by/2\pi e\alpha)$. By the Riemann-von Mangoldt formula for the number of the zeros $1/2+i\gamma(\psi)$ in $0<\gamma(\psi)\le t$, we get first

$$\sum_{Y<\gamma(\psi)\le T} e^{if(\gamma(\psi))}=\frac{1}{2\pi}\log\frac{k}{2\pi}\int_Y^T e^{if(y)}dy+\frac{1}{2\pi}\int_Y^T e^{if(y)}\log y\,dy$$
$$-i\int_Y^T f'(y)\cos(f(y))S(y,\psi)dy$$
$$+\int_Y^T f'(y)\sin(f(y))S(y,\psi)dy+O\Big(\frac{\log kT}{\log\log kT}\Big)$$
$$=S_1+S_2+S_3+S_4+O\Big(\frac{\log kT}{\log\log kT}\Big),\ \text{say},$$

where $S(y,\psi)=(1/\pi)\arg L(1/2+iy,\psi)$ as usual.

By pp. 149–150 of [4], we get

$$S_1+S_2\ll \mathrm{Min}\Big(\frac{1}{|\log(bY/2\pi\alpha)|},\ \sqrt{\alpha}+1\Big)\log kT.$$

We put $\delta=1/(b\log(bT/2\pi\alpha+9))$. By pp. 143–144 of [4], we get

$$S_3=-i\Im\Big\{\frac{b}{2\pi}\int_Y^T\log\frac{bt}{2\pi\alpha}e^{if(t)}\Big(\frac{bt}{2\pi\alpha}\Big)^{b(1/2+\delta)}\log L(1+\delta+it,\psi)dt\Big\}$$
$$+O(B(b,k,\alpha,T)),$$

where we may notice that S_7, S_8 and S_{11} in [4] can be $=O(B(b,k,\alpha,T))$. Treating S_4 similarly, we get

$$S_3+S_4=-\frac{b}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)\psi(n)}{n^{1+\delta}\log n}\int_Y^T\log\frac{bt}{2\pi\alpha}e^{if(t)}\Big(\frac{bt}{2\pi\alpha}\Big)^{b(1/2+\delta)}e^{-it\log n}dt$$
$$+O(B(b,k,\alpha,T))$$
$$=S(T,Y)+O(B(b,k,\alpha,T)),\ \text{say}.$$

To evaluate $S(T,Y)$ is the main problem. We have used before the method of stationary phase over the shorter intervals as in Hardy-Littlewood [6]. The key idea of the present improvement is to use it for the sufficiently longer intervals as in Levinson [7]. The following lemmas can be proved in a similar manner as in Levinson [7] and Gonek [5]. r in Lemmas 1, 2 and 3 will be taken to be $(2\pi\alpha/b)n^{1/b}$ later. We put $\omega=b(1/2+\delta)$.

Lemma 1. *There is a small $c>0$ such that for large br*

$$\int_{r(1-c)}^{r(1+c)}\exp\Big[ibt\log\frac{t}{er}\Big]\Big(\frac{bt}{2\pi\alpha}\Big)^{\omega}dt=e^{-ibr}\Big(\frac{b}{2\pi}\Big)^{\omega-1/2}\frac{r^{\omega+1/2}}{\alpha^{\omega}}e^{(\pi/4)i}+O\Big(\Big(\frac{r}{\alpha}\Big)^{\omega}\Big).$$

Proof. By a change of variable $bt=x$, the left hand side is

$$=\frac{1}{b\alpha^{\omega}}\int_{br(1-c)}^{br(1+c)}\exp\Big[ix\log\frac{x}{e(br)}\Big]\Big(\frac{x}{2\pi}\Big)^{\omega}dx.$$

By Lemma 3.3 of Levinson [7] (and also Lemma 1 of Gonek [5]), this is

$$=(2\pi)^{1/2-\omega}\frac{(br)^{\omega+1/2}}{b\alpha^{\omega}}e^{-ibr}e^{(\pi/4)i}+O\Big(\frac{r^{\omega}}{\alpha^{\omega}}\Big)$$

for large br.

Lemma 2. *For large A and $A<r\le B\le 2A$,*

$$\int_A^B\exp\Big[ibt\log\frac{t}{er}\Big]\Big(\frac{bt}{2\pi\alpha}\Big)^{\omega}dt=e^{-ibr}\Big(\frac{b}{2\pi}\Big)^{\omega-1/2}\frac{r^{\omega+1/2}}{\alpha^{\omega}}e^{(\pi/4)i}+E(r,A,B),$$

where

$$E(r,A,B)=O\left(\left(\frac{A}{\alpha}\right)^{\omega}\right)+O\left(\frac{A^{\omega+1}}{\alpha^{\omega}(|r-A|+\sqrt{A/b})}\right)+O\left(\frac{B^{\omega+1}}{\alpha^{\omega}(|r-B|+\sqrt{B/b})}\right).$$

For $r\le A$ *or* $r>B$,

$$\int_A^B \exp\left[ibt\log\frac{t}{er}\right]\left(\frac{bt}{2\pi\alpha}\right)^{\omega}dt=E(r,A,B).$$

Proof. By a change of variable $bt=x$, we get first that the above integral is

$$=\frac{1}{b\alpha^{\omega}}\int_{bA}^{bB}\exp\left[ix\log\frac{x}{e(br)}\right]\left(\frac{x}{2\pi}\right)^{\omega}dx.$$

Then in the same manner as in the proof of Lemma 3.4 of Levinson [7] (or the proof of Lemma 2 of Gonek [5]), we get our conclusion.

Lemma 3. *For large* A *and* $A<r\le B\le 2A\ll T$,

$$\int_A^B \exp\left[ibt\log\frac{t}{er}\right]\left(\frac{bt}{2\pi\alpha}\right)^{\omega}\log\frac{bt}{2\pi\alpha}dt$$

$$=e^{-ibr}\left(\frac{b}{2\pi}\right)^{\omega-1/2}\frac{r^{\omega+1/2}}{\alpha^{\omega}}e^{(\pi/4)i}\log\frac{br}{2\pi\alpha}+E(r,A,B)\log\frac{T}{\alpha},$$

while for $r\le A$ *or* $r>B$,

$$\int_A^B \exp\left[ibt\log\frac{t}{er}\right]\left(\frac{bt}{2\pi\alpha}\right)^{\omega}\log\frac{bt}{2\pi\alpha}dt=E(r,A,B)\log\frac{T}{\alpha}.$$

Proof. Integration by parts and Lemma 2 gives Lemma 3 as in the proof of Lemma 3.5 of Levinson [7].

Lemma 4. *For large* A *and* $A<B\le 2A\ll T$ *and* $a=1+\delta$,

$$U\equiv\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^a\log n}E\left(\frac{2\pi\alpha n^{1/b}}{b},A,B\right)$$

$$\ll\left(\frac{A}{\alpha}\right)^{b/2}\log\log\frac{T}{2\pi\alpha}+\eta_a(A)\left(\frac{\alpha}{A}\right)^{b/2}\sqrt{A}.$$

Proof.

$$U\ll\frac{A^{\omega}}{\alpha^{\omega}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^a\log n}+\frac{A^{\omega+1}}{\alpha^{\omega}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^a\log n}\frac{1}{|(2\pi\alpha n^{1/b})/b-A|+\sqrt{A/b}}$$

$$+\frac{B^{\omega+1}}{\alpha^{\omega}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^a\log n}\frac{1}{|(2\pi\alpha n^{1/b})/b-B|+\sqrt{B/b}}$$

$$=U_1+U_2+U_3,\ \text{say}.$$

$$U_1\ll\frac{A^{\omega}}{\alpha^{\omega}}\log\log\frac{T}{2\pi\alpha}.$$

$$U_2\ll\frac{A^{\omega+1}}{\alpha^{\omega}}\sum_{\substack{n=2\\ |(2\pi\alpha n^{1/b})/b-A|\le\sqrt{A/b}}}^{\infty}\frac{\Lambda(n)}{n^a\log n}\frac{1}{|(2\pi\alpha n^{1/b})/b-A|+\sqrt{A/b}}$$

$$+\frac{A^{\omega+1}}{\alpha^{\omega}}\sum_{\substack{n=2\\ |(2\pi\alpha n^{1/b})/b-A|>\sqrt{A/b}}}^{\infty}\frac{\Lambda(n)}{n^a\log n}\frac{1}{|(2\pi\alpha n^{1/b})/b-A|+\sqrt{A/b}}$$

$$=\frac{A^{\omega+1}}{\alpha^{\omega}}U_4+\frac{A^{\omega+1}}{\alpha^{\omega}}U_5,\ \text{say}.$$

In U_4, the summation is over

$$\left(\frac{b}{2\pi\alpha}\right)^b\left(A-\sqrt{\frac{A}{b}}\right)^b\leq n\leq\left(\frac{b}{2\pi\alpha}\right)^b\left(A+\sqrt{\frac{A}{b}}\right)^b.$$

Thus if $\alpha\geq CA$, then $U_4=0$. Thus we get

$$U_4\ll\eta_\alpha(A)\frac{\alpha^{b(1+\delta)}}{A^{b(1+\delta)+1/2}}\sum_{(b/2\pi\alpha)^b(A-\sqrt{A/b})^b\leq n\leq(b/2\pi\alpha)^b(A+\sqrt{A/b})^b}\cdot 1$$

$$\ll\eta_\alpha(A)\frac{\alpha^{b(1+\delta)}}{A^{b(1+\delta)+1/2}}(\alpha^{-b}A^{b-1/2}+1)\ll\eta_\alpha(A)\left(\frac{1}{A}+\frac{\alpha^b}{A^{b+1/2}}\right).$$

$$U_5\ll\sum_{n>(b/2\pi\alpha)^b(A+\sqrt{A/b})^b}\frac{\Lambda(n)}{n^a\log n((2\pi\alpha n^{1/b})/b-A+\sqrt{A/b})}$$

$$+\sum_{n<(b/2\pi\alpha)^b(A-\sqrt{A/b})^b}\frac{\Lambda(n)}{n^a\log n(A-(2\pi\alpha n^{1/b})/b+\sqrt{A/b})}$$

$$=U_6+U_7,\ \text{say}.$$

$$U_6\ll\sum_{n\geq(2A)^b(b/2\pi\alpha)^b}\frac{\Lambda(n)}{An^a\log n}$$

$$+\sum_{(b/2\pi\alpha)^b(A+\sqrt{A/b})^b\leq n\leq(2A)^b(b/2\pi\alpha)^b}\frac{\Lambda(n)}{n^a\log n((2\pi\alpha n^{1/b})/b-A+\sqrt{A/b})}$$

$$\ll\frac{1}{A}\log\log\frac{T}{\alpha}+\frac{\eta_\alpha(A)}{A\log(A/\alpha)}\sum_{(b/2\pi\alpha)^b(A+\sqrt{A/b})^b\leq n\leq(2A)^b(b/2\pi\alpha)^b}$$

$$\times\frac{\Lambda(n)}{(n-(bA/2\pi\alpha)^b)+(A/\alpha)^b/\sqrt{A}}$$

$$\ll\frac{1}{A}\log\log\frac{T}{\alpha}+\eta_\alpha(A)\frac{1}{A}\log\log\frac{A}{\alpha}+\eta_\alpha(A)\frac{\alpha^b}{A^{b+1/2}}.$$

Treating U_7 similarly, we get

$$U_2\ll\left(\frac{A}{\alpha}\right)^{b/2}\log\log\frac{T}{2\pi\alpha}+\eta_\alpha(A)\left(\frac{\alpha}{A}\right)^{b/2}\sqrt{A}.$$

Similarly, we get

$$U_3\ll\left(\frac{A}{\alpha}\right)^{b/2}\log\log\frac{T}{2\pi\alpha}+\eta_\alpha(A)\left(\frac{\alpha}{A}\right)^{b/2}\sqrt{A}.$$

These give our Lemma.

Using Lemmas 3 and 4, we get

$$S\left(T,\frac{T}{2}\right)=-e^{(\pi/4)i}\frac{\sqrt{\alpha}}{b}\sum_{(1/2(Tb/2\pi\alpha))^b\leq n\leq(Tb/2\pi\alpha)^b}\frac{\Lambda(n)\psi(n)}{n^{1/2-1/2b}}e^{-2\pi i\alpha n^{1/b}}$$

$$+O\left(\left(\frac{T/2}{\alpha}\right)^{b/2}\log\log\frac{T}{2\pi\alpha}\cdot\log\right)\Big)+O\left(\eta_\alpha(T/2)\left(\frac{\alpha}{T/2}\right)^{b/2}\sqrt{T/2}\log\frac{T}{\alpha}\right).$$

Suppose that an integer L satisfies

$$Y\leq\frac{T}{2^L}<2Y.$$

Then

$$S(T,Y)=-e^{(\pi/4)i}\frac{\sqrt{\alpha}}{b}\sum_{(Yb/2\pi\alpha)^b\leq n\leq(Tb/2\pi\alpha)^b}\frac{\Lambda(n)\psi(n)}{n^{1/2-1/2b}}e^{-2\pi i\alpha n^{1/b}}$$

$$+O\left(\sum_{1\leq l\leq L}\left(\frac{T/2^l}{\alpha}\right)^{b/2}\log\log\frac{T}{2\pi\alpha}\log\frac{T}{\alpha}\right)$$

$$+O\Big(\sum_{1\le l\le L}\eta_\alpha(T/2^l)\Big(\frac{\alpha}{T/2^l}\Big)^{b/2}\sqrt{T/2^l}\log\frac{T}{\alpha}\Big).$$

When $\alpha<CY$, then

$$\sum_{1\le l\le L}\eta_\alpha(T/2^l)\Big(\frac{\alpha}{T/2^l}\Big)^{b/2}\sqrt{T/2^l}\ll\sum_{1\le l\le L}\Big(\frac{\alpha}{T/2^l}\Big)^{b/2}\sqrt{T/2^l}$$

$$\ll\begin{cases}\alpha^{b/2}Y^{-b/2+1/2} & \text{if } b>1\\ \alpha^{b/2}\log T/Y & \text{if } b=1\\ \alpha^{b/2}T^{1/2-b/2} & \text{if } 0<b<1.\end{cases}$$

When $CY\le\alpha$, then

$$\sum_{1\le l\le L}\eta_\alpha(T/2^l)\Big(\frac{\alpha}{T/2^l}\Big)^{b/2}\sqrt{T/2^l}\ll\sum_{M\le l\le L}\Big(\frac{\alpha}{T/2^l}\Big)^{b/2}\sqrt{T/2^l}$$

$$\ll\begin{cases}\alpha^{b/2}Y^{-b/2+1/2} & \text{if } b>1\\ \alpha^{b/2}\log T/Y & \text{if } b=1\\ \sqrt{\alpha} & \text{if } 0<b<1,\end{cases}$$

where M satisfies $CT/2^M\le\alpha\le CT/2^{M-1}$.

Combining all of our estimates, we get our conclusion as described in the introduction.

Acknowledgment. The present work has been done during the author's stay at SFB 170 in Göttingen, Germeny. The author wishes to express his gratitude to Professor U. Christian and SFB for their hospitality.

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