31. Retractive Nil-extensions of Regular Semigroups. II

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Abstract: This paper is the continuation of [6]. Here we consider in particular retractive nil-extensions of unions of groups.

By Theorem 1, some criterions for a semigroup to be a retractive nil-extension of a union of groups are given. A characterization of retractive nil-extensions of semilattice of left and right groups (mixed property) is given by Theorem 2. For the related results see [2] and [5].

Throughout this paper, Z^+ will denote the set of all positive integers. A semigroup S is *π-regular*, if for every $a \in S$ there exists $n \in Z^+$ such that $a^n \in a^n S a^n$. Let us denote by $Reg(S)$ ($Gr(S)$, $E(S)$) the set of all regular (completely regular, idempotent) elements of a semigroup S. A semigroup S is *Archimedean*, if for all $a, b \in S$ there exists $n \in Z^+$ such that $a^n \in SbS$. A semigroup S is *completely Archimedean*, if S is Archimedean and has a primitive idempotent (or, equivalently, if it is a nil-extension of a completely simple semigroup [1]). If e is an idempotent of a semigroup S, then by G_e we denote the maximal subgroup of S with e as its identity and $T_e = \{a \in S \mid (\exists n \in Z^+) a^n \in G_e\}$. For undefined notions and notations we refer [1], [10] and [6].

Veronesi's theorem [11]. *A semigroup S is a semilattice of completely Archimedean semigroups, if and only if S is π-regular and $Reg(S) = Gr(S)$.*

Munn's lemma [9]. *Let a be an element of a semigroup S such that a^n lies in some subgroup G of S for some $n \in Z^+$. If e is an identity of G, then $ea = ae \in G_e$ and $a^m \in G_e$ for all $m \in Z^+$, $m \geq n$.*

Lemma 1 [5]. *Let S be a nil-extension of a union of groups K. Then every retraction φ of S onto K has the following representation:*

$$\varphi(x) = xe \qquad \text{if } x \in T_e,\ e \in E(S).$$

Theorem 1. *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a union of groups;*

(ii) *S is π-regular and for all $x, a, y \in S$ there exists $n \in Z^+$ such that*

$$xa^n y \in x^2 S y^2; \qquad (1)$$

(iii) *S is a subdirect product of a union of groups and a nil-semigroup.*

Proof. (i)$\Rightarrow$(ii). Let S be a retractive nil-extension of a union of groups K, with the retraction φ of S onto K. Let $x, a, y \in S$. Then there exists $n \in Z^+$ such that $a^n \in K$, so $xa^n y \in K$. Let $x^m \in G_e$, $y^k \in G_f$, $m, k \in Z^+$,

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$e, f \in E(S)$. By Lemma 1, it follows that $\varphi(x)=xe=xx^m u \in x^2S$, for some $u \in G_e$, and in a similar way we prove that $\varphi(y) \in Sy^2$. Thus

$$xa^n y=\varphi(xa^n y)=\varphi(x)a^n\varphi(y) \in x^2SSSy^2 \subseteq x^2Sy^2,$$

since $xa^n y \in K$, so (1) holds. It is clear that S is π-regular.

(ii)$\Rightarrow$(i). Let S be π-regular and let (1) hold. Let $a \in Reg(S)$. Then $a=axa$ for some $x \in S$, so

$$\begin{aligned} a &= a(xa)^n xa && \text{for all } n \in \boldsymbol{Z}^+, \\ &\in a^2S(xa)^2 && \text{by (1)}, \\ &\subseteq a^2S. \end{aligned}$$

In a similar way we can prove that $a \in Sa^2$. Hence, $Reg(S)=Gr(S)$, so by Veronesi's theorem it follows that S is a semilattice Y of completely Archimedean semigroups S_α, $\alpha \in Y$. Also, for every $\alpha \in Y$, S_α is a nil-extension of a completely simple semigroup K_α. Let $x \in S$, $e \in E(S)$. By (1) if follows that

$$\begin{aligned} xe &= xee^n e && \text{for all } n \in \boldsymbol{Z}^+, \\ &= (xe)^2 u && \text{for some } u \in S, \text{ by (1)}, \end{aligned}$$

whence $xe=(xe)^{m+1}u^m$ for every $m \in \boldsymbol{Z}^+$. In a similar way we can prove that there exists $v \in S$ such that $ex=v^m(ex)^{m+1}$ for all $m \in \boldsymbol{Z}^+$. Assume that $xe \in S_\alpha$ for some $\alpha \in Y$. Then it is easy to verify that $xeu^m \in S_\alpha$ for all $m \in \boldsymbol{Z}^+$. Let $m \in \boldsymbol{Z}^+$ be such that $(xe)^m \in K_\alpha$. Then

$$xe=(xe)^m(xe)u^m \in K_\alpha S_\alpha \subseteq K_\alpha \subseteq Reg(S).$$

Hence, $xe \in Reg(S)$. Similarly we can prove that $ex \in Reg(S)$. Therefore $K=Reg(S)=Gr(S)$ is an ideal of S.

Assume that $xe \in x^m Se$, i.e. $xe=x^m ue$ for some $u \in S$. By (1) we obtain that there exists $n \in \boldsymbol{Z}^+$ such that $x^m(ue)^n e \in x^{2m}Se$. Since K is a completely regular ideal of S, we have $ue \in K$ and $ue\mathcal{H}(ue)^n$, where $\mathcal{H}$ is the Green's H-relation on K, so there exists $v \in S$ such that $ue=(ue)^n v$. Thus

$$xe=x^m ue=x^m uee=x^m(ue)^n ve=x^m(ue)^n eve \in x^{2m}Seve \subseteq x^{m+1}Se.$$

Now by induction we obtain

$$(2) \qquad xe \in x^m Se, \qquad \text{for every } m \in \boldsymbol{Z}^+.$$

Similarly we can prove that

$$(3) \qquad ex \in eSx^m, \qquad \text{for every } m \in \boldsymbol{Z}^+.$$

Define a mapping $\varphi: S \rightarrow K$ by $\varphi(x)=xe$, if $x \in T_e$, $e \in E(S)$. Let $x, y \in S$. Then $x \in T_e$, $y \in T_f$, $xy \in T_g$ for some $e, f, g \in E(S)$, i.e. $x^n \in G_e$, $y^m \in G_f$, $(xy)^k \in G_g$ for some $n, m, k \in \boldsymbol{Z}^+$. By (2) and (3) we obtain $yg \in y^m Sg = fy^m Sg$, $xf \in x^n Sf = ex^n Sf$, $ey \in eSy^m = eSy^m f$, and $exy \in eS(xy)^k = eS(xy)^k g$, whence

$$yg=fyg, \quad xf=exf, \quad ey=eyf, \quad exy=exyg.$$

By this and by Munn's lemma it follows that

$$\varphi(xy)=xyg=xfyg=exfyg=exyg=exy=xey=xeyf=\varphi(x)\varphi(y).$$

Therefore, φ is a retraction, so S is a retractive nil-extension of a union of groups.

(i)$\Leftrightarrow$(iii). This follows from Theorem 1 [6].

Corollary 1 [7]. *The following conditions on a semigroup S are equi-*

valent:

(i) *S is an n-inflation of a union of groups*;

(ii) *for all $x, y \in S$, $xS^{n-1}y \subseteq x^2S^ny^2$ ($xy \in x^2Sy^2$, if $n=1$)*;

(iii) *S is a subdirect product of a union of groups and an $(n+1)$-nilpotent semigroup.*

Theorem 2. *A semigroup S is a retractive nil-extension of a semilattice of left and right groups if and only if S is π-regular and for all $x, a, y \in S$ there exists $n \in Z^+$ such that*

$$xa^ny \in x^2Sy^2x \cup yx^2Sy^2. \tag{4}$$

Proof. Let S be a retractive nil-extension of a semigroup K and let K be a semilattice Y of left and right groups K_α, $\alpha \in Y$, with a retraction φ of S onto K. Let $x, a, y \in S$. Then there exists $n \in Z^+$ such that $a^n \in K$. As in the proof of Theorem 1, we obtain that $xa^ny \in x^2Sy^2$. On the other hand, since xa^ny, a^ny^2x, $yx^2a^n \in K$, we then have that xa^ny, a^ny^2x, $yx^2a^n \in K_\alpha$ for some $\alpha \in Y$, so by Lemma 1.1 [8] it follows that

$$xa^ny \in xa^nyK_\alpha a^ny^2x \subseteq x^2Sy^2Sy^2x \subseteq x^2Sy^2x,$$

if K_α is a left group, or

$$xa^ny \in yx^2a^nK_\alpha xa^ny \subseteq yx^2Sx^2Sy^2 \subseteq yx^2Sy^2,$$

if K_α is a right group. Therefore, (4) holds. It is clear that S is π-regular.

Conversely, let S be π-regular and let (4) hold. Let $a, b \in S$. Then there exists $n \in Z^+$ such that

$$(ab)^{n+1} = a(ba)^nb \in a^2Sb^2a \cup ba^2Sb^2 \subseteq Sa \cup bS,$$

so by Theorem 3.1 [3] we obtain that S is a semilattice Y of semigroups S_α, $\alpha \in Y$, and for every $\alpha \in Y$, S_α is a nil-extension of a left or a right group K_α. Let $x \in S$, $e \in E(S)$. Then by (4) it follows that

$$\begin{aligned} xe &= (xe)e^ne \qquad\qquad \text{for every } n \in Z^+ \\ &\in (xe)^2Sexe \cup e(xe)^2Se \\ &\subseteq (xe)^2S \cup e(xe)^2Se. \end{aligned}$$

Let $xe \in e(xe)^2S$. Then $xe = exe$, so $xe \in e(xe)^2S = (xe)^2S$. Therefore, $xe = (xe)^2u$ for some $u \in S$, whence $xe = (xe)^{m+1}u^m$ for every $m \in Z^+$. Similarly we can show that there exists $v \in S$ such that $ex = v^m(ex)^{m+1}$ for every $m \in Z^+$. Now, as in the proof of Theorem 1 we can prove that $K = Reg(S) = Gr(S)$ is an ideal of S. It is clear that K is a semilattice of left and right groups. Now we shall prove that

$$xe \in x^mSe \qquad \text{for every } m \in Z^+. \tag{5}$$

First, assume that $xe = exe$. Then it is easy to verify that $(xe)^m = x^me$ for all $m \in Z^+$. Since $xe \in K$, $xe\mathcal{H}(xe)^m = x^me$, where $\mathcal{H}$ is a Green's H-relation on K, so (5) holds. Assume that $xe \neq exe$ and assume that $xe = x^mue$ for some $u \in S$ and $m \in Z^+$. Then by (4) it follows that there exists $n \in Z^+$ such that $x^m(ue)^ne \in x^{2m}Sex^m \cup ex^{2m}Se$. Moreover, since $ue \in K$ and $ue\mathcal{H}(ue)^n$, there exists $v \in K$ such that $ue = (ue)^nv = (ue)^nev$. Thus

$$xe = x^mue = x^m(ue)^neve \in x^{2m}Sex^m \cup ex^{2m}Se.$$

Since $xe \neq exe$, $xe \in x^{2m}Sex^m$, so $xe \in x^{m+1}Se$. Whence by induction we obtain that (5) holds. Similarly we can show that

(6) $$ex \in eSx^m \quad \text{for every } m \in \boldsymbol{Z}^+,$$
and as in the proof of Theorem 1 we obtain that K is a retract of S.

Corollary 2. *A semigroup S is an n-inflation of a semilattice of left and right groups if and only if $xS^{n-1}y \subseteq x^2S^ny^2x \cup yx^2S^ny^2$ ($xy \in x^2Sy^2x \cup yx^2Sy^2$, if $n=1$) for all $x, y \in S$.*

Theorem 3. *A semigroup S is a nil-extension of a semilattice of left groups if and only if S is π-regular and for all $x, a, y \in S$ there exists $n \in \boldsymbol{Z}^+$ such that*

(7) $$xa^ny \in xSx.$$

Proof. Let S be a nil-extension of a semigroup K which is a semilattice of left groups. Then S is a semilattice Y of completely Archimedean semigroups $S_\alpha, \alpha \in Y$. Let $K_\alpha = S_\alpha \cap K$, $\alpha \in Y$. Then it is clear that K_α is a left group for every $\alpha \in Y$. Let $x, a, y \in S$. Then there exists $n \in \boldsymbol{Z}^+$ such that $a^n \in K$, so $xa^ny, ya^nx \in K_\alpha$ for some $\alpha \in Y$. Now by Lemma 1.1 [8] we obtain

$$xa^ny \in xa^nyK_\alpha ya^nx \subseteq xSx.$$

Therefore, (7) holds. It is clear that S is π-regular.

Conversely, let S be π-regular and let (7) hold. Let $x \in S$, $e \in E(S)$. Then

$$\begin{aligned} xe &= xee^ne && \text{for every } n \in \boldsymbol{Z}^+, \\ &\in xeSxe && \text{by (7)}, \end{aligned}$$

so $xe \in Reg(S)$. By this it follows that $Reg(S)$ is a left ideal of S. Moreover,

$$\begin{aligned} ex &= ee^nx && \text{for every } n \in \boldsymbol{Z}^+, \\ &\in eSe && \text{by (7)}, \end{aligned}$$

so $ex = exe$, whence

$$\begin{aligned} ex &= exe^ne && \text{for every } n \in \boldsymbol{Z}^+, \\ &\in exSex && \text{by (7)}. \end{aligned}$$

Therefore, $ex \in Reg(S)$, so $Reg(S)$ is a right ideal of S. Therefore, S is a nil-extension of a regular semigroup $K = Reg(S)$.

Let $a, b \in K$. Then there exists $e \in E(K)$ such that $ae = a$, whence

$$\begin{aligned} ab &= ae^nb && \text{for every } n \in \boldsymbol{Z}^+, \\ &\in aSa && \text{by (7)}, \\ &\subseteq Ka && \text{since } K \text{ is an ideal of } S, \end{aligned}$$

so by Theorem IV 3.10. [10] it follows that K is a semilattice of left groups.

Theorem 4. *A semigroup S is a retractive nil-extension of a semilattice of left groups if and only if S is π-regular and for all $x, a, y \in S$ there exists $n \in \boldsymbol{Z}^+$ such that*

(8) $$xa^ny \in x^2Sx.$$

Proof. Let S be a retractive nil-extension of a semilattice of left groups. Then it is clear that S is π-regular and in a similar way as in the proof of Theorem 2 we can show that (8) holds.

Conversely, let S be π-regular and let (8) hold. Then by Theorem 3 we see that S is a nil-extension of a semigroup K and that K is a semilattice of left groups. Let $x \in S$, $e \in E(S)$. As in the proof of Theorem 1 we can show that $xe \in x^m S$ for every $m \in Z^+$. Moreover, by (8) it follows that $ex = exe$, so $(ex)^m = ex^m$ for every $m \in Z^+$, and since $ex \mathcal{H} (ex)^m = ex^m$ (where $\mathcal{H}$ is a Green's H-relation on K), $ex \in Sx^m$ for every $m \in Z^+$. Now, as in the proof of Theorem 1 we see that the mapping $\varphi: S \to K$ defined by $\varphi(x) = xe$, if $x \in T_e$, $e \in E(S)$, is a retraction. Therefore, S is a retractive nil-extension of a semilattice of left groups.

Corollary 3. *A semigroup S is an n-inflation of a semilattice of left groups if and only if $xS^{n-1}y \subseteq x^2S^nx$ ($xy \in x^2Sx$, if $n=1$) for all $x, y \in S$.*

Theorem 5. (i) *A semigroup S is a retractive nil-extension of a completely simple semigroup if and only if S is π-regular, Archimedean and for all $a, b \in S$ there exists $n \in Z^+$ such that $(ab)^n \in a^2Sb^2$.*

(ii) *A semigroup S is a retractive nil-extension of a left group if and only if S is π-regular, Archimedean and for all $a, b \in S$ there exists $n \in Z^+$ such that $(ab)^n \in a^2Sa$.*

Proof. (i) Let S be a retractive nil-extension of a completely simple semigroup K and let φ be a retraction of S onto K. As in the proof of Theorem 1 we can prove that $\varphi(x) \in x^2S \cap Sx^2$ for all $x \in S$, and since for all $a, b \in S$ there exists $n \in Z^+$ such that $(ab)^n \in K$, we then have

$$(ab)^n = \varphi((ab)^n) = (\varphi(a)\varphi(b))^n \in a^2Sb^2.$$

The converse follows from Theorem 1 [4].

(ii) We can prove this similarly as (i).

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