56. Scattering Theory for Semilinear Wave Equations with Small Data in Two Space Dimensions

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1. Introduction. We consider a scattering problem for the semilinear wave equation

(1) $$u_{tt} - \Delta u = F(u), \qquad (x, t) \in R^n \times R,$$

where $F(u) = \lambda |u|^p$ or $\lambda |u|^{p-1}u$, $\lambda \in R$, $p > 1$ and $n = 2$.

We compare the asymptotic bahavior as $t \to \pm\infty$ of solution of (1) with those of suitable solutions of the Cauchy problem for the free wave equation

(2) $$u_{tt} - \Delta u = 0, \qquad (x, t) \in R^n \times R,$$

in the sense of energy norm.

Among many theorems on the construction of the scattering operator for (1), we mention a remarkable result of Pecher [17] which has improved the previous results by Strauss [20], Klainerman [10], Mochizuki and Motai [15, 16].

Pecher has shown that in three space dimensions ($n = 3$), the scattering operator for (1) exists for regular and small data if $p > 1 + \sqrt{2}$. This lower bound of p is optimal, since John [8] showed that there exist global C^2-solutions to the Cauchy problem for the equation $u_{tt} - \Delta u = |u|^p$ with smooth and small data of compact support if $p > 1 + \sqrt{2}$, and it was also shown by [8, 9] and Schaeffer [19] that the solution blows up in finite time for small data of compact support if $1 < p \leq 1 + \sqrt{2}$.

In two space dimensions, Strauss [20] proved that the scattering operator exists if $p > 2 + \sqrt{5}$, provided the smooth data are small. This result was improved to $p > 2 + \sqrt{3}$ by Klainerman [10], Mochizuki and Motai [15, 16]. However, these results are not sharp, since Glassey [7] proved that the Cauchy problem for $u_{tt} - \Delta u = |u|^p$ has global C^2-solutions if $p > \frac{3 + \sqrt{17}}{2}$, provided the smooth data of compact support are small, and moreover it was shown by Glassey [6] and Schaeffer [19] that if $1 < p \leq \frac{3 + \sqrt{17}}{2}$, global solutions do not exist, provided the data of compact support satisfy a certain condition.

Our aim of this paper is to prove that if $p > \frac{3 + \sqrt{17}}{2}$, the scattering operator exists for smooth and small data in two space dimensions.

We note that in the global existence theorems by John and Glassey, the data are compactly supported. Since we solve the Cauchy problem for (1) with the data given at $t = -\infty$, the assumption of compactly-supported data is not natural. In this connection, we briefly state the previous results concerning the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-1}u, & (x, t) \in R^n \times (0, \infty), \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x), & x \in R^n, \quad (n = 2,3), \end{cases}$$

with small data of non compact support.

Asakura [2], Kubota [13] and the author [21-23] have shown that global solutions exist for small data $f(x) \in C^3(R^n)$, $g(x) \in C^2(R^n)$ satisfying

$$D_x^\alpha f(x),\ D_x^\beta g(x) = O(|x|^{-1-k}) \text{ as } |x| \to \infty,\ |\alpha| \leq 3,\ |\beta| \leq 2$$

if $k \geq \frac{2}{p-1}$, $p > 1 + \sqrt{2}$ for $n = 3$ and $p > \frac{3+\sqrt{17}}{2}$ for $n = 2$. The result due to Pecher [17] also corresponds to the removal of compact support assumption for initial data in three space dimensions.

In contrast to this existence theorem, Asakura [2], Agemi and Takamura [1], the author [21] have proved that if the data satisfy

$$f(x) = 0,\ g(x) \geq \frac{\varepsilon}{(1+|x|)^{1+k}}, \quad \varepsilon > 0$$

and $0 < k < \frac{2}{p-1}$, then the solution blows up in finite time even with

$$p > \begin{cases} 1 + \sqrt{2} & (n = 3) \\ \dfrac{3+\sqrt{17}}{2} & (n = 2). \end{cases}$$

The construction of the scattering operator for small data is almost equivalent to the construction of global solution for the Cauchy problem of (1) with small initial data given at $t = 0$ in the case of three space dimensions, but the construction of the scattering operator for small data does not follow directly from the proofs in [7], [13], [21] and [23] concerning the global existence of solutions for the Cauchy problem of the above equation with small initial data given at $t = 0$ in two space dimensions. Because we have to consider the integral equation with unbounded integral region associated to the above equation:

$$(3) \quad u(x, t) = u_0^-(x, t) + \frac{1}{2\pi}\int_{-\infty}^{t}\int_{|x-y|\leq t-s} \frac{F(u(y, s))}{\sqrt{(t-s)^2 - |x-y|^2}}\, dyds, \quad \text{for } t \in R,$$

where $u_0^-(x, t)$ is a solution of $u_{tt} - \Delta u = 0$ which $u(x, t)$ approaches asymptotically as $t \to -\infty$. The proof of the basic estimate for the above integral equation is more difficult and complicated than that for the Cauchy problem of $u_{tt} - \Delta u = |u|^{p-1}u$ in two space dimensions. The basic estimate can be proved following Kovalyov [11, 12] and Asakura [2].

2. Assumptions and theorem. We study the scattering problem for the semilinear wave equation (1) with $n = 2$. We make the following hypothesis.

(H1) $F(u) \in C^2(R)$ and there exist $p > \frac{3+\sqrt{17}}{2}$, $\lambda > 0$ such that

$$|F^{(j)}(u)| \leq \lambda |u|^{p-j} \quad \text{for } |u| \leq 1,\ j = 0, 1, 2,$$

$$|F''(u) - F''(v)| \leq \lambda |\phi|^{p-3}|u - v| \quad \text{for } |u|, |v| \leq 1,\ \phi = \max\{|u|, |v|\}.$$

Let $u_0^-(x,t)$ be the C^2-solution of the Cauchy problem

(4) $$u_{tt} - \Delta u = 0 \qquad (x,t) \in R^2 \times R,$$

(5) $$u(x,0) = f(x),\ u_t(x,0) = g(x) \qquad x \in R^2.$$

The hypothesis concerning the data is the following:

(H2) $f(x) \in C^3(R^2)$, $g(x) \in C^2(R^2)$ satisfy

$$\sum_{|\alpha| \le 3} |D_x^\alpha f(x)| + \sum_{|\beta| \le 2} |D_x^\beta g(x)| \le \frac{\varepsilon}{(1+|x|)^{1+k}}$$

with $k > 0$, where ε is a small parameter.

Remark 1. (H2) implies $f(x) \in H^2(R^2)$ and $g(x) \in H^1(R^2)$.

We consider to solve the integral equation (3) in the space X_k, $k > \frac{2}{p-1}$, where

$$X_k = \{u(x,t) : D_x^\alpha u(x,t) \in C(R^2 \times R) \text{ for } |\alpha| \le 2,\ \|u\|_{X_k} < \infty\},$$

$$\|u\|_{X_k} = \sum_{|\alpha| \le 2} \|D_x^\alpha u\|_k,$$

$$\|u\|_k = \begin{cases} \sup_{\substack{x\in R^2\\ t\in R}} \dfrac{(1+|t|+|x|)^{\frac{1}{2}}(1+||t|-|x||)^m}{\ln\left(2+||t|-|x||\right)} |u(x,t)| & \left(k > \frac{1}{2}\right), \\ \sup_{\substack{x\in R^2\\ t\in R}} (1+|t|+|x|)^{\frac{1}{2}} \left(1 + \ln \dfrac{1+|t|+|x|}{1+||t|-|x||}\right)^{-1} |u(x,t)| & \left(k = \frac{1}{2}\right), \\ \sup_{\substack{x\in R^2\\ t\in R}} (1+|t|+|x|)^k |u(x,t)| & \left(0 < k < \frac{1}{2}\right), \end{cases}$$

and

$$m = \min\left(\frac{1}{2}, \frac{p-3}{2}, k - \frac{1}{2}\right).$$

We put

$$\|u(t)\|_e = \{\|D_x u(t)\|_{L^2(R^2)}^2 + \|\partial_t u(t)\|_{L^2(R^2)}^2\}^{\frac{1}{2}}.$$

Theorem 2. *Assume the hypotheses* (H1) *and* (H2).

(i) *If* $k > \frac{2}{p-1}$ *and* ε *is sufficiently small, depending on* λ, p *and* k, *then there exists a unique* C^2*-solution* $u(x,t)$ *of the integral equation* (3) *such that*

$$\|u(t) - u_0^-(t)\|_e \le \frac{C}{(1+|t|)^l} \to 0 \quad (t \to -\infty),$$

where C *is a constant depending on* λ, p, k *and*

$$l = \begin{cases} m & \left(k > \frac{1}{2}\right), \\ k & \left(0 < k \le \frac{1}{2}\right). \end{cases}$$

(ii) *Moreover, there exists a unique* C^2*-solution* $u_0^+(x,t)$ *of* (4) *such that for the solution* $u(x,t)$ *of* (3) *given by part* (i),

$$\|u(t) - u_0^+(t)\|_e \le \frac{C}{(1+|t|)^l} \to 0 \quad (t \to +\infty).$$

Thus, we can define the scattering operator

$$S : (u_0^-(0),\ \partial_t u_0^-(0)) \to (u_0^+(0),\ \partial_t u_0^+(0))$$

for the equation (1).

Remark 3. The solution $u(x, t)$ of (3) given by Theorem 2 is a solution of (1) (see [2], [7], [8], [17]).

Remark 4. Recently, K. Kubota and K. Mochizuki [14] have obtained independently a similar result using different methods from ours. However our decay estimates of solutions of (1) are slightly sharper than [14].

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References

[1] R. Agemi and H. Takamura: The lifespan of classical solutions to nonlinear wave equations in two space dimensions (preprint).
[2] F. Asakura: Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimensions. Comm. Partial Differential Equations, **11**, no. 13, 1459–1487 (1986).
[3] J. Ginibre and G. Velo: Conformal invariance and time decay for non linear wave equations. I. Ann. Inst. Henri Poincaré, Phys. Théor., **47**, 221–261 (1987).
[4] ——: Conformal invariance and time decay for non linear wave equations. II. ibid., **47**, 263–276 (1987).
[5] ——: Scattering theory in the energy space for a class of non-linear wave equations. Commun. Math. Phys., **123**, 535–573 (1989).
[6] R. T. Glassey: Finite-time blow-up for solutions of nonlinear wave equations. Math. Z., **177**, 323–340 (1981).
[7] ——: Existence in the large for $\Box u = F(u)$ in two space dimensions. ibid., **178**, 233–261 (1981).
[8] F. John: Blow-up of solutions of nonlinear wave equations in three space dimensions. Manuscripta Math., **28**, 235–268 (1979).
[9] ——: Nonlinear wave equations, Formation of singularities. Lehigh University, University Lecture Series, Amer. Math. Soc., Providence (1990).
[10] S. Klainerman: Long-time behavior of solutions to nonlinear evolution equations. Arch. Rat. Mech. Analy., **78**, 73–98 (1982).
[11] M. Kovalyov: Long-time bahaviour of solutions of a system of nonlinear wave equations. Comm. Partial Differential Equations, **12**, no. 5, 471–501 (1987).
[12] ——: Long-time existence of solutions of nonlinear wave equations. Ph. D. Thesis. Courant Institute of Mathematical Sciences, New York University (1986).
[13] K. Kubota: Existence of a global solution to a semi-linear wave equation with initial data of non-compact support in low space dimensions (preprint).
[14] K. Kubota and K. Mochizuki: On small data scattering for 2-dimensional semilinear wave equations (preprint).
[15] K. Mochizuki and T. Motai: The scattering theory for the nonlinear wave equation with small data. J. Math. Kyoto Univ., **25**, 703–715 (1985).
[16] ——: The scattering theory for the nonlinear wave equation with small data. II. Publ. RIMS, Kyoto Univ., **23**, 771–790 (1987).
[17] H. Pecher: Scattering for semilinear wave equations with small data in three

space dimensions. Math. Z., **198**, 277–289 (1988).

[18] H. Pecher: Global smooth solutions to a class of semilinear wave equations with strong nonlinearities. Manuscripta Math., **69**, 71–92 (1990).

[19] J. Schaeffer: The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p. Proc. Roy. Soc. Edinburgh, **101A**, 31–44 (1985).

[20] W. A. Strauss: Nonlinear scattering theory at low energy. J. Funct. Anal., **41**, 110–133 (1981).

[21] K. Tsutaya: Global existence theorem for semilinear wave equations with non compact data in two space dimensions (to appear in J. Diff. Eq.).

[22] ——: Global existence and the life span of solutions of semilinear wave equations with data of non compact support in three space dimensions (preprint).

[23] ——: A global existence theorem for semilinear wave equations with data of non compact support in two space dimensions (to appear in Comm. Partial Differential Equations).