55. *Evolution Governed by "Generalized" Dissipative Operators*

By Yoshikazu KOBAYASHI*) and Naoki TANAKA**)

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Let X be a real Banach space with norm $|\cdot|$. For $x \in X$ and $f \in L^1(0, T : X)$, we consider the abstract Cauchy problem of the form

$$(\mathrm{CP}\,; x, f) \qquad \begin{cases} (d/dt)u(t) \in Au(t) + f(t), & \text{for } t \in (0, T), \\ u(0) = x, \end{cases}$$

where A is a multivalued operator in X satisfying the dissipative condition of the following general type:

(GD) There exists a "uniqueness function" ω such that

$$[x_1 - x_2, y_1 - y_2]_- \leq \omega(|x_1 - x_2|)$$

for $x_1, x_2 \in D(A)$, $y_1 \in Ax_1$ and $y_2 \in Ax_2$.

We mean by the "uniqueness function" a real valued continuous function ω defined on $[0, \infty)$ such that $\omega(0) = 0$ and that $r(t) \equiv 0$ is the unique solution of the initial value problem: $(d/dt)r(t) = \omega(r(t))$, $t \geq 0$ and $r(0) = 0$. The semi-inner products $[\cdot, \cdot]_\pm$ are defined by

$$[x, y]_+ = \lim_{\lambda \downarrow 0}(|x + \lambda y| - |x|)/\lambda \text{ and } [x, y]_- = \lim_{\lambda \uparrow 0}(|x + \lambda y| - |x|)/\lambda$$

for $x, y \in X$.

The first aim of this note is to introduce a notion of generalized solutions, i.e., that of mild solutions, to the Cauchy problem $(\mathrm{CP}\,; x, f)$ and to investigate its fundamental properties. The second is to discuss the existence of mild solutions of the problem $(\mathrm{CP}\,; x, f)$. Here, we sketch our results. The details of the results will be exhibited elsewhere.

1. Properties of mild solutions. We introduce a notion of solutions, called herein mild solutions, which refers directly to the approximation method used to establish the existence of solutions, so-called *method of discretization in time.*

Definition 1. Let $\varepsilon > 0$. A piecewise constant function $u : [0, t_N] \to X$ is said to be an *ε-approximate solution* of $(\mathrm{CP}\,; x, f)$ on $[0, T]$, if there exists a partition $\{0 = t_0 < t_1 < \cdots < t_N\}$ of the interval $[0, t_N]$ and a finite sequence $((x_i, f_i) : i = 1, \cdots, N)$ with the four properties below:

$$(\varepsilon.1) \qquad u(t) = \begin{cases} x_0 & \text{for } t = 0 \\ x_1 & \text{for } t \in (t_{i-1}, t_i] \end{cases}$$

and

$$(t_i - t_{i-1})^{-1}(x_i - x_{i-1}) \in Ax_i + f_i,$$

for $i = 1, \cdots, N$,

*) Department of Applied Mathematics, Faculty of Engineering, Niigata University.

**) Department of Mathematics, Faculty of Science, Kochi University.

(ε.2) $t_i - t_{i-1} \leq \varepsilon,\ i = 1,2,\cdots,N$ and $T - \varepsilon < t_N \leq T$,

(ε.3) $|x_0 - x| \leq \varepsilon$,

(ε.4) $\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} |f(t) - f_i|\, dt \leq \varepsilon$.

Definition 2. A continuous function $u : [0, T] \to X$ is said to be a *mild solution* of (CP ; x, f) on $[0, T]$, provided that for each $\varepsilon > 0$ there is an ε-approximate solution u^ε of (CP ; x, f) on $[0, T]$ such that $|u(t) - u^\varepsilon(t)| \leq \varepsilon$ for t in the domain of u^ε.

We have the following type of uniqueness theorem for mild solutions (cf. Bénilan [1]).

Theorem 1. *Let $u : [0, T] \to X$ and $v : [0, T] \to X$ be mild solutions of* (CP ; x, f) *and* (CP ; y, g) *on* $[0, T]$, *respectively. Then*

$$|u(t) - v(t)| - |u(s) - v(s)| \leq \int_s^t \{[u(\sigma) - v(\sigma), f(\sigma) - g(\sigma)]_+ + \omega(|u(\sigma) - v(\sigma)|)\}\, d\sigma$$

for $s, t \in [0, T]$ with $s \leq t$. If $f = g$ in $L^1(0, T : X)$ and $x = y$ in particular, then $u(t) = v(t)$ on $[0, T]$.

2. Existence of mild solutions. The following is one of the most fundamental theorems concerning the convergence of ε-approximate solutions (cf. Kobayashi [6] and Takahashi [10]).

Theorem 2. *For each $\varepsilon > 0$, let $u^\varepsilon : [0, t^\varepsilon_{N_\varepsilon}] \to X$ be an ε-approximate solution of* (CP ; x, f) *on* $[0, T]$. *If $x \in \overline{D(A)}$, then the following statements are mutually equivalent.*

(i) $\sup\{|u^\varepsilon(t)| : t \in [0, t^\varepsilon_{N_\varepsilon}]\}$ *is bounded as* $\varepsilon \downarrow 0$.

(ii) *There exists a mild solution u of* (CP ; x, f) *on* $[0, T]$ *such that* $\sup\{|u^\varepsilon(t) - u(t)| : t \in [0, t^\varepsilon_{N_\varepsilon}]\}$ *converges to zero as* $\varepsilon \downarrow 0$.

The proof of Theorem 2 is based on

Lemma 1. *Suppose that for $\lambda, \mu > 0$, three sequences $\{t^\lambda_i\}_{i=0}^{N_\lambda}$, $\{t^\mu_j\}_{j=0}^{N_\mu}$ and $\{a^{\lambda,\mu}_{i,j} : i = 0,1,\cdots,N_\lambda$ and $j = 0,1,\cdots,N_\mu\}$ of nonnegative numbers satisfy the following four conditions :*

(i) $0 = t^\lambda_0 < t^\lambda_1 < \cdots < t^\lambda_{N_\lambda}$, $\quad 0 = t^\mu_0 < t^\mu_1 < \cdots < t^\mu_{N_\mu}$,

$h^\lambda_i := t^\lambda_i - t^\lambda_{i-1} \leq \lambda$, $\ i = 1,2,\cdots,N_\lambda$, $\ T - \lambda < t^\lambda_{N_\lambda} \leq T$,

$h^\mu_j := t^\mu_j - t^\mu_{j-1} \leq \mu$, $\ j = 1,2,\cdots,N_\mu$, $\ T - \mu < t^\mu_{N_\mu} \leq T$,

(ii) *there exists a number $K > 0$ such that $a^{\lambda,\mu}_{i,j} \leq K$ for $0 \leq i \leq N_\lambda$ and $0 \leq j \leq N_\mu$,*

(iii) *there exist $M \geq 0$ and $A_{\lambda,\mu} \geq 0$ with $\limsup_{\lambda,\mu\downarrow 0} A_{\lambda,\mu} \leq A < \infty$ such that*

$$a^{\lambda,\mu}_{i,j} \leq A_{\lambda,\mu} + |t^\lambda_i - t^\mu_j|\, M \text{ for } i = 0 \text{ or } j = 0,$$

(iv) *there exist $L \geq 0$ and $B_{\lambda,\mu} \geq 0$ with $\limsup_{\lambda,\mu\downarrow 0} B_{\lambda,\mu} \leq B < \infty$ such that*

$$(a^{\lambda,\mu}_{i,j} - a^{\lambda,\mu}_{i-1,j})/h^\lambda_i + (a^{\lambda,\mu}_{i,j} - a^{\lambda,\mu}_{i,j-1})/h^\mu_j \leq \omega(a^{\lambda,\mu}_{i,j}) + L\,|t^\lambda_i - t^\mu_j| + B_{\lambda,\mu}$$

for $1 \leq i \leq N_\lambda$ and $1 \leq i \leq N_\mu$.

Then we have

$$\limsup_{\lambda,\mu\downarrow 0}(\sup\{a^{\lambda,\mu}_{i,j} : 0 \leq i \leq N_\lambda \text{ and } 0 \leq j \leq N_\mu \text{ with } |t^\lambda_i - t^\mu_j| \leq \lambda + \mu + h\})$$

$$\leq \sup\{m_K(t:\eta+hM, Lh+\eta): t\in[0,T]\}+\eta^{-1}K(A+B)$$

for $h\geq 0$ *and* $\eta>0$. *Here* $m_K(t:\alpha,\beta)$ *is the nonextentable maximal solution of the initial problem*: $(d/dt)r(t)=\omega_K(r(t))+\beta$, $t\geq 0$ *and* $r(0)=\alpha$, *where* $\alpha,\beta\geq 0$ *and* ω_K *is defined by*

$$\omega_K(r)=\begin{cases}\omega(r) & \text{for } 0\leq r\leq K,\\ \omega(K) & \text{for } r\geq K.\end{cases}$$

Roughly speaking, Lemma 1 is proved by comparing $a_{i,j}^{\lambda,\mu}$ with $A_{K,\eta}(t_i^\lambda, t_j^\mu)$, by using the following difference approximate version (Lemma 2) of comparison theorem, where $A_{K,\eta}(t,s):=m_K(t\wedge s:\eta+|t-s|M, L|t-s|+\eta)$ is a solution of partial differential equation:

$$\begin{cases}u_t(t,s)+u_s(t,s)=\omega_K(u(t,s))+L|t-s|+\eta, \\ \qquad\qquad\qquad\qquad\text{in } D'((0,T)\times(0,T))\\ u(t,s)=\eta+|t-s|M,\ t=0 \text{ or } s=0.\end{cases}$$

Lemma 2. *Suppose that* $\lambda>0$ *and two families* $\{a,a_0,b_0\}$ *and* $\{A,A_0,B_0\}$ *of nonnegative numbers satisfy two inequalities*

$$(A-A_0)/\lambda\geq\omega_K(A)+B_0 \text{ and } (a-a_0)/\lambda\leq\omega_K(a)+b_0.$$

If $b_0+2\rho_{\omega_K}(\lambda(\omega_0(K)+b_0))<B_0$ *then* $a_0\leq A_0$ *implies* $a\leq A$. *Here* ρ_{ω_K} *is the modulus of continuity of* ω_K *and* $\omega_0(K)=\sup\{|\omega_K(r)|: r\geq 0\}$.

Outline of the proof "(i) $\Rightarrow$ (ii)" of Theorem 2: Let $\lambda,\mu>0$ and $g\in \mathrm{Lip}([0,T]:X)$. We set

$$A_{i,j}^{\lambda,\mu}(g):=\max\{|x_i^\lambda-x_j^\mu|-E_i^\lambda(g)-E_j^\mu(g), 0\}$$

for $0\leq i\leq N_\lambda$ and $0\leq j\leq N_\mu$, where $E_i^\lambda(g)$ and $E_j^\mu(g)$ are defined by

$$E_i^\lambda(g)=\sum_{k=1}^{i}(t_k^\lambda-t_{k-1}^\lambda)|f_k^\lambda-g(t_k^\lambda)|,\ i=0,1,\cdots,N_\lambda$$

and

$$E_j^\mu(g)=\sum_{k=1}^{j}(t_k^\mu-t_{k-1}^\mu)|f_k^\mu-g(t_k^\mu)|,\ j=0,1,\cdots,N_\mu.$$

Then we may show that $\{A_{i,j}^{\lambda,\mu}: 0\leq i\leq N_\lambda \text{ and } 0\leq j\leq N_\mu\}$ satisfies three estimates corresponding to (ii), (iii) and (iv) of Lemma 1. Since $|x_i^\lambda-x_j^\mu|\leq A_{i,j}^{\lambda,\mu}(g)+E_{N_\lambda}^\lambda(g)+E_{N_\mu}^\mu(g)$, it may be proved by Lemma 1 that

$$\limsup_{\lambda,\mu\downarrow 0}(\sup\{|u^\lambda(t)-u^\mu(t)|: t\in[0,t_{N_\lambda}^\lambda]\cap[0,t_{N_\mu}^\mu]\})$$
$$\leq\sup\{m_K(t:\eta,\eta): t\in[0,T]\}+2\int_0^T|f(t)-g(t)|\,dt$$
$$+\eta^{-1}K\Big(2|x-u|+\rho\omega_K\Big(2\int_0^T|f(t)-g(t)|\,dt\Big)\Big)$$

for $\eta>0$, $u\in D(A)$ and $g\in\mathrm{Lip}([0,T]:X)$. It thus is shown that the limit $\lim_{\lambda\downarrow 0}u^\lambda(t)$ exists for $t\in[0,T)$ by noting that $m_K(t:\eta,\eta)$ converges to zero uniformly on $[0,T]$ as $\eta\to 0+$ and $\mathrm{Lip}([0,T]:X)$ is dense in $L^1(0,T:X)$.

We define $S(A)=\{z\in X: \liminf_{\lambda\to 0+}\lambda^{-1}d(R(I-\lambda A), x+\lambda z)=0$ for any $x\in\overline{D(A)}\}$. By using Theorem 2 and [11, Lemma 3.2], we have the following existence theorem of mild solutions (cf. Bénilan [1] and Kobayashi [6]):

Theorem 3. *Suppose that the uniqueness function* ω *satisfies the condition that* $\limsup_{r\to\infty}\omega(r)/r<\infty$. *If* $x\in\overline{D(A)}$ *and* $f(t)\in S(A)$ *for almost all*

$t \in (0, T)$, *then there exists a* (*unique*) *mild solution of* (CP ; x, f) *on* $[0, T]$.

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