11. *Higher Specht Polynomials for the Symmetric Group*

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§0. Introduction. We are concerning with constructing a basis of the S_n-module $H = \boldsymbol{Q}[x_1,\dots,x_n]/(e_1,\dots,e_n)$, where $(e_1,\dots,e_n)$ denotes the ideal generated by elementary symmetric polynomials $e_j = e_j(x_1,\dots,x_n)$ for $j = 1,\dots,n$.

Let $P = \boldsymbol{Q}[x_1,\dots,x_n]$ be the algebra of polynomials of n variables $x_1,\dots,x_n$ with rational coefficients, on which the symmetric group S_n acts by the permutation of the variables:

$$(\sigma f)(x_1,\dots,x_n) = f(x_{\sigma(1)},\dots,x_{\sigma(n)}) \quad (\sigma \in S_n).$$

Let us denote by Λ the subalgebra of P consisting of the symmetric polynomials. Let $e_j(x_1,\dots,x_n) = \sum_{1\le i_1<\dots<i_j\le n} x_{i_1}\dots x_{i_j}$ be the elementary symmetric polynomial of degree j and put $J_+ = (e_1,\dots,e_n)$, an ideal generated by $e_1,\dots,e_n$. The quotient algebra $H = P/J_+$ has a structure of an S_n-module. It is well known that the S_n-module H is isomorphic to the regular representation. In other words, every irreducible representation of S_n occurs in H with multiplicity equal to its dimension. We will give a combinatorial procedure to obtain a basis of each irreducible component of H.

For a Young diagram λ of n cells, one can construct an S_n-module $V(\lambda)$ as follows (cf. [5]). For a tableau T of shape λ put

$$\Delta_T = \prod_{\beta\ge 1} \Delta_T(\beta) \in P,$$

where $\Delta_T(\beta)$ is the product of differences $x_i - x_j$ for the pair $\{(i, j)\,;\, i<j\}$ appearing in the β-th column in T. The polynomial Δ_T is called the Specht polynomial of T. The space $V(\lambda)$ spanned by all the Specht polynomials Δ_T for tableaux T of shape λ is naturally equipped with a structure of an S_n-module. It is well known that $V(\lambda)$ is irreducible for any Young diagram λ and has a basis $\{\Delta_T\,;\, T \text{ is a standard tableau of shape } \lambda\}$.

Our basis of H is parametrized by the pair of standard tableaux (S, T) of the same shape and turns out to be a natural generalization of these standard Specht polynomials. One finds a related topic in [1].

§1. Standard tableaux and their indices. Fix a Young diagram $\lambda = (\lambda_1,\dots,\lambda_n)\,(\lambda_1 \ge \dots \ge \lambda_n \ge 0)$ consisting of n cells. We often say that λ is a partition of n and write $\lambda \vdash n$. The set of tableaux (resp. standard tableaux) of shape λ is denoted by $Tab(\lambda)$ (resp. $STab(\lambda)$) (cf. [5]). For a standard tableau S of shape λ, one can associate the index tableau $i(S)$ of the same shape in the following manner (cf. [2]). Define the word $w(S)$ by reading S from the bottom to the top in consecutive columns, starting from the left. The number 1 in the word $w(S)$ has index 0. If the number k in the word has index p, then $k+1$ has index p or $p+1$ according as it lies to the right or the left of k. The charge $c(S)$ of S is defined to be the sum of the indices. For example, if $S = \begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 \end{smallmatrix}$, then $w(S) = 3_1 1_0 5_2 2_0 4_1$ where the indices are attached as the subscript. Filling the index in corresponding cell of the given tableau S, we obtain the index tableau $i(S)$ of S. Note that one can recover S by knowing $i(S)$. Let S be a standard tableau and T a tableau of the same shape λ, and let $c(\alpha, \beta)$ be the number in the (α, β)-cell of T. Put $x_T^{i(S)} = x_1^{i_1} \cdots x_n^{i_n}$. Here i_k is the index in the (α, β)-cell of $i(S)$ where $k = c(\alpha, \beta)$. For example take $S = \begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 \end{smallmatrix}$ and $T = \begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 \end{smallmatrix}$, so that $i(S) = \begin{smallmatrix} 0 & 0 & 1 \\ 1 & 2 \end{smallmatrix}$. Then $x_T^{i(S)} = x_1^0 x_2^1 x_3^0 x_4^2 x_5^1$. In the next section, we will define higher Specht polynomials by using monomials $x_T^{i(S)}$.

§2. Higher Specht polynomials. For $T \in Tab(\lambda)$, let $R(T)$ and $C(T)$ denote the row stabilizer and the column stabilizer of T respectively and consider the Young symmetrizer

$$\varepsilon_T = \sum_{\sigma \in R(T)} \sum_{\tau \in C(T)} (sgn\tau)\tau\sigma,$$

which is an element of the group algebra $\boldsymbol{Q}S_n$. We now define the polynomial F_T^S by

$$F_T^S(x_1, \ldots, x_n) = \varepsilon_T(x_T^{i(S)}),$$

for $S \in STab(\lambda)$ and $T \in Tab(\lambda)$. For the canonical standard tableau S_0 of shape λ, where the cells are numbered from the left to the right in consecutive rows, starting from the top, $F_T^{S_0}$ is proportional to the Specht polynomial of T. We will call F_T^S the higher Specht polynomial associated with (S, T). For a standard tableau $T \in STab(\lambda)$ the higher Specht polynomial F_T^S is said to be standard. The Robinson-Schensted correspondence assures that

$$\sum_{\lambda \vdash n} |\, STab(\lambda) \,|^2 = n!.$$

Hence we have the set of $n!$ standard higher Specht polynomials $\mathscr{F} = \{F_T^S; S, T \in STab(\lambda), \lambda \vdash n\}$. The following theorem is a fundamental property of the standard higher Specht polynomials.

Theorem 1. (1) *The set $\mathscr{F}$ gives a free basis of Λ-module P.*
(2) *The set $\mathscr{F}$ gives a free basis of $\boldsymbol{Q}$-algebra H.*

Here we only give an outline of the proof. Consider a Λ-valued symmetric Λ-bilinear form on P:

$$\langle f, g \rangle = \sum_{\sigma \in S_n} (sgn\sigma)\, \sigma(fg) / \prod_{i<j} (x_i - x_j), \quad f, g \in P.$$

This bilinear form is nothing but the divided difference $\partial_{\sigma_0}(fg)$ corresponding to the longest element $\sigma_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$ in S_n (cf. [3]). To prove that $\mathscr{F}$ is a free Λ-basis of P, it is sufficient to see that the Gramian with respect to the bilinear form $\langle\ ,\ \rangle$ is a non-zero constant. First of all it is not difficult to check that for all $f, g \in P$,

$$\langle \varepsilon_{T_1}(f), \varepsilon_{T_2}(g) \rangle = 0 \quad \text{or} \quad \langle \varepsilon_{T_2}(f), \varepsilon_{T_1}(g) \rangle = 0,$$

unless $T_2 = T_1'$, where T' denotes the transposed tableau of T. To show that, $\langle \varepsilon_T(x_T^{i(S)}), \varepsilon_{T'}(x_{T'}^{i(S')}) \rangle$ is a non-zero constant for $S, T \in STab(\lambda)$, it suffices to check the following

Lemma. *A pair* $(\sigma, \tau) \in R(T) \times C(T)$ *satisfies*

$$\langle \sigma(x_T^{i(S)}), \tau(x_{T'}^{i(S')}) \rangle \neq 0,$$

if and only if σ *fixes* $i(S)$ *and* τ *fixes* $i(S')'$.

If the set of indices of S_1 does not coincide with that of S_2, then we see that

$$\langle \varepsilon_T(x_T^{i(S_1)}), \varepsilon_{T'}(x_{T'}^{i(S_2')}) \rangle = 0 \quad \text{or} \quad \langle \varepsilon_T(x_T^{i(S_2)}), \varepsilon_{T'}(x_{T'}^{i(S_1')}) \rangle = 0,$$

for any T. It happens that the sets of indices of S_1 and S_2 coincide for the distinct $S_1, S_2 \in STab(\lambda)$. For example, both $S_1 = \begin{matrix} 1 & 2 & 6 \\ 3 & 4 & \\ 5 & & \end{matrix}$ and $S_2 = \begin{matrix} 1 & 2 & 4 \\ 3 & 6 & \\ 5 & & \end{matrix}$ have the indices $\{0,0,1,1,2,2\}$. In this case we can prove the existence of a total ordering "$<$" in the subset of $STab(\lambda)$ consisting of such tableaux, for which, if $S_1 < S_2$, then

$$\langle \sigma(x_T^{i(S_1)}), \tau(x_{T'}^{i(S_2')}) \rangle = 0,$$

for all $T \in STab(\lambda)$ and for all $\sigma \in R(T)$, $\tau \in C(T)$. All these arguments imply that the Gramian of $\mathscr{F}$ with respect to $\langle\, , \rangle$ is a non-zero constant. The statement (2) is an easy consequence of (1).

§3. Irreducible representations in H. For $\lambda \vdash n$, let $V(\lambda)$ be the Specht module corresponding to λ, which is spanned by $\{\varepsilon_T(x_T^{i(S_0)})\,;\, T \in Tab(\lambda)\}$, where S_0 is the canonical standard tableau of shape λ. As is well known, $V(\lambda)$ is irreducible and has a basis $\{\varepsilon_T(x_T^{i(S_0)})\,;\, T \in STab(\lambda)\}$. In particular, we know

$$\dim V(\lambda) = |\, STab(\lambda)\,| = \frac{n!}{\prod_{(\alpha,\beta)} h(\alpha, \beta)},$$

where $h(\alpha, \beta)$ denotes the hook length of the (α, β)-cell in the Young diagram λ. Since the S_n-module H is isomorphic to the regular representation, each irreducible representation occurs in H with multiplicity equal to its dimension. According to the graduation $H = \bigoplus_{d \geq 0} H_d$ the multiplicity in H_d is described by the following Poincaré series:

$$M_\lambda(q) = \frac{q^{n(\lambda)} \prod_{k=1}^{n}(1 - q^k)}{\prod_{(\alpha,\beta)}(1 - q^{h(\alpha,\beta)})},$$

where $n(\lambda) = \sum_{i=1}^{n} (i-1)\lambda_i$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash n$. In other words, the irreducible representation isomorphic to $V(\lambda)$ occurs m_d times in H_d, where $M_\lambda(q) = \sum_{d \geq 0} m_d q^d$. It is known that $M_{\lambda'}(q)$ is the Kostka-Foulkes polynomial of shape λ and weight (1^n) (cf. [1,2]).

A basis of each irreducible component is given by higher Specht polynomials as follows.

Theorem 2. *Fix* $\lambda \vdash n$ *and* $S \in STab(\lambda)$. *Then the space* $V^S(\lambda) = \sum_{T \in Tab(\lambda)} \boldsymbol{Q} F_T^S$ *is an irreducible* S_n*-module in* $H_{c(S)}$ *isomorphic to* $V(\lambda)$ *equipped with a basis* $\mathscr{F}^S(\lambda) = \{F_T^S\,;\, T \in STab(\lambda)\}$.

To prove this theorem it suffices to check that the higher Specht polynomials $F_T^S (T \in Tab(\lambda))$ satisfy the following Garnir relations. Take the β-th and the γ-th columns of T with $\beta < \gamma$. Fix a number α_0 so that $1 \leq \alpha_0 \leq$

$a(\gamma)$, where $\alpha(\gamma)$ is the length of the γ-th column. Denote by $S_{\alpha_0}^{\beta,\gamma}$ the group of permutations of the set $\{c(\alpha_0, \beta), c(\alpha_0+1, \beta), \ldots, c(\alpha(\beta), \beta), c(1, \gamma)$ $c(2, \gamma), \ldots, c(\alpha_0, \gamma)\}$ and define the Garnir element by

$$G_{\alpha_0}^{\beta,\gamma} = \sum_{\sigma \in S_{\alpha_0}^{\beta,\gamma}} (sgn\sigma)\sigma \in \mathbf{Q}S_n.$$

The Garnir relations for $F \in P$ read

$$G_{\alpha_0}^{\beta,\gamma}(F) = 0 \quad (1 \leq \alpha_0 \leq \alpha(\gamma), \beta < \gamma).$$

It can be proved according to the line in [4] that $F = F_T^S$ satisfies the Garnir relations for any $S \in STab(\lambda)$ and $T \in Tab(\lambda)$.

Proofs and detailed discussions will be published elsewhere.

References

[1] A. M. Garsia and C. Procesi: On certain graded S_n-modules and the q-Kostka polynomials. Adv. Math., **94**, 82–138 (1992).

[2] I. G. Macdonald: Symmetric Functions and Hall Polynomials. Oxford University Press (1979).

[3] ——: Notes on Schubert Polynomials. Université de Québec à Montréal (1991).

[4] M. H. Peel: Specht modules and symmetric groups. J. Alg., **36**, 88–97 (1975).

[5] B. Sagan: The Symmetric Groups. Wadsworth and Brooks (1991).