10. *Prime Ideals in Noncommutative Valuation Rings in Finite Dimensional Central Simple Algebras*

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1. Introduction. In [2], Dubrovin introduced a notion of non-commutative valuation rings in simple Artinian rings, and proved some elementary properties of them. He obtained in [3] more detailed results concerning valuation rings in finite dimensional central simple algebras over fields.

In this paper, we investigate prime ideals in non-commutative valuation rings in the case of algebras. The key result is Proposition 9 which states that, for any ideal A of a valuation ring R, $\cap A^n$ is a prime ideal of R. Using this result, we characterize branched and unbranched prime ideals.

2. Throughout this paper, let V be a valuation domain with the quotient field K, and let R be a valuation ring in the sense of [2] in a finite dimensional central simple K-algebra Σ with its center V and $KR = \Sigma$.

First, we shall list the elementary properties of a non-commutative valuation ring R which are used frequently.

(A) R-ideals are linearly ordered by inclusion and the Jacobson radical $J(R)$ is the unique maximal ideal of R (§2 Theorem 4 (1) and §1 Theorem 4 of [2]).

(B) Each overring S of R is also a valuation ring, and $J(S)$ is a prime ideal of R (Theorem 4 (2) of [2, §2]).

(C) For any R-ideal A, $O_r(A) = O_l(A)$, where $O_r(A) = \{q \in \Sigma \mid Aq \subseteq A\}$, the *right order* of A, and $O_l(A) = \{q \in \Sigma \mid qA \subseteq A\}$, the *left order* of A (Corollary to Proposition 4 of [3, §2]).

(D) For any non-zero element $x \in R$, there is some regular $z \in R$ such that $RxR = zT = Tz$, where $T = O_r(RxR) = O_l(RxR)$ (Proposition 3 of [3, §2]).

(E) For any prime ideal P of R, $C(P) = \{c \in R \mid [c+P]$ is regular mod $P\}$ is a regular Ore set of R and so there exists the localization of R with respect to $C(P)$. We denote this by R_P. Let $p = P \cap V$. Then we have $R_P = R_p$, where R_p denotes the localization of R with respect to $V - p$ (Theorem 1 of [3, §2]).

(F) The mapping $P \to R_P$ is an inclusion reversing bijection between the set of prime ideals of R and the set of overrings of R. The inverse mapping is $S \to J(S)$. (Corollary to Theorem 4 of [2, §2] and Theorem 1 (3) of [3, §2].)

Now we shall investigate prime ideals of R. For any ideal A of R, we

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define $\sqrt{A} = \cap \{P: \text{prime ideal of } R \mid P \supseteq A\}$, the radical of A. The following is trivial from Property (A).

Lemma 1. *If A is an ideal of R, then $\sqrt{A}$ is a prime ideal of R.*

Lemma 2. *Let P be a prime ideal of R. Then $R_P = O_r(P) = O_l(P)$.*

Proof. First we note that $O_r(P) = O_l(P)$ by Property (C). From Property (F), we have $P = J(R_p)$, and so P is an ideal of R_P. Hence we have $R_P \subseteq O_r(P)$. Conversely, since $O_r(P) \supseteq R \supseteq P$ and P is an ideal of $O_r(P)$, we have $R \supseteq J(O_r(P)) \supseteq P$ by Properties (B) and (A). It follows from Properties (E) and (F) that $R_P = R_p \supseteq R_{J(O_r(P)) \cap V} = R_{J(O_r(P))} = O_r(P)$. Hence $R_P = O_r(P)$.

From Lemma 2, we have

Lemma 3. *Let P_1 and P_2 be prime ideals of R such that $P_1 \subseteq P_2$. Then we have $O_r(P_1) \supseteq O_r(P_2)$.*

An ideal Q of R is called a *primary ideal if* $xRy \subseteq Q$ and $x \notin Q$, then $y \in \sqrt{Q}$, and if $xRy \subseteq Q$ and $x \notin \sqrt{Q}$, then $y \in Q$. If $\sqrt{Q} = P$, then we say that Q is a *P-primary ideal.* A prime ideal P of R is said to be *branched* if there exists a P-primary ideal Q of R such that $Q \neq P$. In other case, P is called an *unbranched* prime ideal.

Lemma 4. *For any ideal A of R, we have $O_r(A) \subseteq O_r(\sqrt{A})$. If A is a primary ideal, then the equality holds.*

Proof. By property (A), we have $O_r(A) \supseteq O_r(\sqrt{A})$ or $O_r(A) \subseteq O_r(\sqrt{A})$. Assume that $O_r(A) \supset O_r(\sqrt{A})$. Then we have $J(O_r(A)) \subset J(O_r(\sqrt{A})) = J(R_{\sqrt{A}}) = \sqrt{A}$ by Property (F), Lemmas 1 and 2. On the other hand, by Property (C), A is an ideal of $O_r(A)$, and so we have $A \subseteq J(O_r(A))$ by Property (A). It follows that $\sqrt{A} \subseteq J(O_r(A))$, since, by Property (B), $J(O_r(A))$ is a prime ideal of R. Thus we have $\sqrt{A} \subseteq J(O_r(A)) \subset J(O_r(\sqrt{A})) = \sqrt{A}$, a contradiction. Hence $O_r(A) \subseteq O_r(\sqrt{A})$. Next assume that A is a primary ideal, and let $P = \sqrt{A}$ and $p = P \cap V$. For any element $q = ac^{-1} \in A_p$, where $a \in A$ and $c \in V - p$, we have $qRc = qcR = aR \subseteq A$. On the other hand, $c \notin P$ and $q \in A_p \subseteq P_p = P \subseteq R$. Since A is a P-primary ideal, we have $q \in A$ and so $A_p = A$, that is, A is an ideal of R_p. Hence, by Lemma 2, we have $O_r(\sqrt{A}) = R_p \subseteq O_r(A)$.

Remark 5. The equality in Lemma 4 does not hold in general. For example, let V be a valuation domain with rank 2, and let $0 \neq P_1 \subset P_2$ be prime ideals of V. Then by Theorem 17.3 (e) of [5, p. 190], P_1 and P_2 are branched, and so $P_1 = \sqrt{aV}$ for some $a(\neq 0) \in V$. Assume that $O_r(aV) = O_r(\sqrt{aV})(= R_{P_1})$. Then aV is an ideal of R_{P_1}, and so aV is a P_1-primary ideal by the next Lemma 6. Hence, by Theorem 17.3 (a) of [5], we have $aV \cdot xV = aV$ for any $x \in V - P_1$, and so $xV = V$, that is, x is a unit of V. It follows that P_1 is a maximal ideal of V, a contradiction. Thus we have $O_r(aV) \subset O_r(\sqrt{aV})$.

Lemma 6. *Let $Q \subseteq P$ be ideals of R and assume that P is prime. Then the following are equivalent.*

(1) *Q is a P-primary ideal.*

(2) *$\sqrt{Q} = P$ and Q is an ideal of R_P.*

Proof. (1) $\Rightarrow$ (2): Assume that Q is a P-primary ideal. Then $\sqrt{Q} = P$ by the definition. By Lemmas 2 and 4, $O_r(Q) = O_r(\sqrt{Q}) = O_r(P) = R_P$, and so Q is an ideal of R_P.
(2) $\Rightarrow$ (1): Assume that the condition (2) holds and $xRy \subseteq Q$, where $x, y \in R$. If $x \notin P$, then we have $R_P x R_P \not\subseteq P$, and so $R_P x R_P = R_P$, because P is the unique maximal ideal of R_P by Properties (A) and (F). Since Q is an ideal of R_P and $R_P = R_p$ by Property (E), where $p = P \cap V$, we have $Q \supseteq R_P x R y = R_P x R_P y = R_P y \ni y$. Similarly, if $y \notin P$, then $x \in Q$. Hence Q is a P-primary ideal.

Corollary 7. *If Q_1 and Q_2 are P-primary ideals of R, then $Q_1 Q_2$ is also a P-primary ideal* (see [5]).

Proof. It is clear that $\sqrt{Q_1 Q_2} = P$ and $Q_1 Q_2$ is an R_P-ideal. Hence it is P-primary by Lemma 6.

Lemma 8. *Let Q be a P-primary ideal of R. Then for any ring T such that $R \subset T \subset R_P$, Q is a P-primary ideal as an ideal of T.*

Proof. First we note that Q is an ideal of T by Lemma 6. Let I', J' be ideals of T, and assume that $J' \not\subseteq P$. If $I'J' \subseteq Q$, then we have $(I' \cap R)(J' \cap R) \subseteq Q$ and $J' \cap R \not\subseteq P$, because $J' = (J' \cap R)R_{J(T)} = (J' \cap R)T$ by Property (F). Hence $I' \cap R \subseteq Q$, and so $I' = (I' \cap R)T \subseteq QT = Q$.

Proposition 9. *For any ideal A of R, $P_0 = \cap A^n$ is a prime ideal.*

Proof. (i) First, we assume that A is a primary ideal. Let x, y be elements of R such that $x \notin P_0$ and $y \notin P_0$. Then $x \notin A^n$ and $y \notin A^m$ for some integers $n, m > 0$, and so $RxR \supset A^n$ and $RyR \supset A^m$ by Property (A). By Property (D), $RxR = z_1 T_1 = T_1 z_1$ and $RyR = z_2 T_2 = T_2 z_2$, where $T_1 = O_r(RxR)$, $T_2 = O_r(RyR)$ and $z_1, z_2 \in R$. Now we have $T_1 \supseteq T_2$ or $T_1 \subseteq T_2$ by Property (A). We may assume that $T_1 \supseteq T_2$. Then $A^{n+m} \subseteq A^n RyR = A^n T_2 z_2 \subseteq z_1 T_1 T_2 z_2$. If $A^n T_2 z_2 = z_1 T_1 T_2 z_2$, then $A^n T_2 = z_1 T_1 T_2 = z_1 T_1$, because z_2 is a regular element. Further we have $\sqrt{A} = \sqrt{A^n} \subseteq \sqrt{RxR}$, and so $O_r(A^n) = O_r(\sqrt{A}) \supseteq O_r(\sqrt{RxR}) \supseteq O_r(RxR) = T_1 \supseteq T_2$ by Corollary 7 and Lemmas 3 and 4. Thus we have $A^n = A^n T_2 = z_1 T_1$, a contradiction. Hence $A^n T_2 z_2 \subset z_1 T_1 T_2 z_2$, and so $A^{n+m} \subset z_1 T_1 T_2 z_2 = RxRyR$, and we have $xRy \not\subseteq P_0$. Thus P_0 is a prime ideal.
(ii) In general case, let $P = \sqrt{A}$. If P is not idempotent, then $A \supset P^k$ for some $k > 0$, because if $A \subseteq P^n$ for all $n > 0$, then $A \subseteq \cap P^n \subset P = \sqrt{A}$, and by case (i), $\cap P^n$ is a prime ideal, a contradiction. Hence $A \not\subseteq P^k$ for some k. Then by Property (A), we have $A \supset P^k$. It follows that $\cap P^n \supseteq \cap A^n \supseteq \cap P^{kn} \supseteq \cap P^n$, and so $P_0 = \cap A^n = \cap P^n$ is a prime ideal by case (i). If P is idempotent and $P = A$, then $P_0 = \cap A^n = \cap P^n = P$ is a prime ideal. If P is idempotent and $P = \sqrt{A} \supset A$, then for any element $x \in P - A$, we have $RxR \not\subseteq A$, and so $A \subset RxR \subseteq R_P x R_P \subseteq P$. If $P = R_P x R_P$, then by Property (D) and Lemma 2, $P = zR_P$ for some $z \in P$, so we have $P^2 \neq P$, a contradiction. Hence $P \supset R_P x R_P$. Put $Q = R_P x R_P$. Then Q is an R_P-ideal and $\sqrt{Q} = P$, because $P \supset Q \supset A$ and $P = \sqrt{A}$. So Q is a P-primary ideal by Lemma 6. Further, if $A \subset Q^n$ for all $n > 0$, then $P \supset \cap Q^n \supseteq A$ and $\cap Q^n$ is a prime ideal by case (i). This is a contradiction, since $P = \sqrt{A}$. Hence

$A \supseteq Q^k$ for some $k > 0$, and so $\cap Q^n \supseteq \cap A^n \supseteq \cap Q^{kn} \supseteq \cap Q^n$. Thus $P_0 = \cap A^n = \cap Q^n$ is a prime ideal.

Lemma 10. *Let A be an ideal of R.*

(1) *If $A^k = A^{k+1}$ for some $k > 0$, then A is an idempotent prime ideal.*

(2) *Let P be a prime ideal of R such that $P \subset A$. Then $P \subseteq \cap A^n$.*

(3) *If B is an ideal of R and $A \subset \sqrt{B}$, then $A^n \subseteq B$ for some $n > 0$.*

Proof. (1) Put $P_0 = \cap A^n$. If $A^k = A^{k+1}$, then $P_0 = A^k$. By Proposition 9, P_0 is a prime ideal. Hence $P_0 \supseteq A$, and so $A = P_0$ is a prime ideal. Further, since $A = A^k$, A is idempotent.

(2) If $A^n \subseteq P$ for some $n > 0$, then $A \subseteq P$ and we have $A \subset A$, a contradiction. Hence $A^n \not\subseteq P$ for any $n > 0$, and so $A^n \supseteq P$ for any $n > 0$ by Property (A). Thus $P \subseteq \cap A^n$.

(3) If $A^n \not\subseteq B$ for any $n > 0$, then $A^n \supset B$ and so $B \subseteq \cap A^n$. Since $\cap A^n$ is a prime ideal, by Proposition 9, it follows that $\sqrt{B} \subseteq \cap A^n \subseteq A$, that is, $A \subset A$, a contradiction. Hence $A^n \subseteq B$ for some $n > 0$.

Lemma 11. *Let $\mathcal{S}$ be a set of prime ideals of R and let $P = \cup_{p' \in \mathcal{S}} P'$. Then*

(1) *$O_r(P) = \cap_{p' \in \mathcal{S}} O_r(P')$.*

(2) *P is a prime ideal.*

Proof. (1) Let $P' \in \mathcal{S}$. Since $\sqrt{P} \supseteq P \supseteq P'$, we have $O_r(P) \subseteq O_r(\sqrt{P}) \subseteq O_r(P')$ by Lemmas 3 and 4. Hence $O_r(P) \subseteq \cap O_r(P')$. Conversely, let $x \in \cap O_r(P')$ and let $a \in P$. Since $a \in P'$ for some $P' \in \mathcal{S}$, it follows that $ax \in P'x \subseteq P' \subseteq P$, and so $Px \subseteq P$. Hence $x \in O_r(P)$.

(2) Let $x, y \notin P$ and let $P' \in \mathcal{S}$. Then $x, y \notin P'$. Hence, by Properties (A) and (D), $P' \subset RxR = z_1T_1 = T_1z_1$ and $P' \subset RyR = z_2T_2 = T_2z_2$, where $T_1 = O_r(RxR)$, $T_2 = O_r(RyR)$ and $z_1, z_2 \in R$. We may assume that $T_1 \supseteq T_2$ by Property (A). Then, since P' is a prime ideal of R, we have $P' \not\supseteq RxR \cdot RyR = z_1T_1T_2z_2 = z_1T_1z_2 = T_1z_1z_2$. Hence $z_1z_2 \notin P'$, because $T_1 = O_r(RxR) \subseteq O_r(\sqrt{RxR}) \subseteq O_r(P')$ by Lemmas 3 and 4. Thus $z_1z_2 \notin P$, and so $xRy \not\subseteq P$.

Now, concerning branched and unbranched ideals, we have the following.

Theorem 12. *Let P be a prime ideal of R, and let $P_0 = \cap P^n$.*

(1) *If P is branched and $P \neq P^2$, then*

(i) *$\{P^k \mid k > 0\}$ is the full set of P-primary ideals of R,*

(ii) *$P = zT = Tz$ for some $z \in P$, where $T = O_r(P)$,*

(iii) *there is no prime ideal P' such that $P \supset P' \supset P_0$ and P_0 is a prime ideal.*

(2) *If P is branched and $P = P^2$, then*

(i) *for any P-primary ideal Q ($\neq P$), $\cap Q^n = \cap \{Q_\lambda \mid Q_\lambda$: P-primary ideal$\}$,*

(ii) *$Q_0 := \cap \{Q_\lambda \mid Q_\lambda : P$-primary ideal$\}$ is a prime ideal,*

(iii) *there is no prime ideal P' such that $P \supset P' \supset Q_0$.*

(iv) *$P = \cup \{Q_\lambda \mid Q_\lambda : P$-primary with $Q_\lambda \neq P\}$.*

(3) *The following are equivalent:*

(i) *P is branched.*

(ii) *$P=\sqrt{A}$ for some ideal $A(\neq P)$.*

(iii) *$P=\sqrt{RaR}$ for some $a\in R$.*

(iv) *P is not the union of prime ideals P' such that $P'\subset P$.*

(v) *There is a prime ideal M such that $M\subset P$ and there are no prime ideals P' such as $M\subset P'\subset P$.*

(4) *P is unbranched if and only if $P=\cup\{P_\lambda \mid P_\lambda(\subset P): \text{prime ideal}\}$.*

Proof. (1) By Corollary 7, P^k is a P-primary ideal for any $k>0$. Conversely, let Q be any P-primary ideal of R. Then we have $P^2\subset P=\sqrt{Q}$. By Lemma 10 (3), we have $(P^2)^n\subseteq Q$ for some integer $n>0$. Let k be the smallest integer such as $P^k\subseteq Q$. Then $P^{k-1}\not\subseteq Q$ and so there is some $y\in P^{k-1}-Q$. From Property (D), there exists $z\in R$ such that $RyR=zT=Tz$, where $T=O_r(RyR)$. As $\sqrt{RyR}=P$, $T\subseteq O_r(\sqrt{RyR})=O_r(Q)$ by Lemma 4. Put $A=Qz^{-1}$. Then $A\subseteq zTz^{-1}=T$, and so A is an ideal of T. On the other hand, $Q=AzT$ and $zT\not\subseteq Q$. It follows from Lemma 8 that $P\supseteq A$. Hence $Q=AzT\subseteq P\,RyR\subseteq P\,P^{k-1}=P^k$, and so $Q=P^k$. Thus (i) is proved. (ii) follows from Lemma 8 of [2, §2]. To prove (iii), let P' by any prime ideal such that $P'\subset P$. Then by Lemma 10 (2), we have $P'\subseteq\cap P^n=P_0$. Hence there is no prime ideal P' between P and P_0, and P_0 is a prime ideal by Proposition 9.

(2) By Corollary 7, Q^n is a P-primary ideal, and so $\cap Q^n\supseteq\cap Q_\lambda$. Conversely, for any P-primary ideal Q_λ, we have $Q\subset P=\sqrt{Q_\lambda}$. By Lemma 10 (3), $Q_\lambda\supseteq Q^n$ for some $n>0$, and so $\cap Q_\lambda\supseteq\cap Q^n$. Hence we have $\cap Q_\lambda=\cap Q^n$. From this fact and Proposition 9, (ii) follows. To prove (iii), let P' be a prime ideal such that $P'\subset P$. Then, for any P-primary ideal Q_λ, we have $Q_\lambda\not\subset P'$, because $\sqrt{Q_\lambda}=P$. Hence $P'\subseteq Q_\lambda$, and so $P'\subseteq\cap Q_\lambda=Q_0$. To prove (iv), let $Q=\cup\{Q_\lambda \mid Q_\lambda: P\text{-primary with } Q_\lambda\neq P\}$. Then it is P-primary by Lemma 6. Assume that $P\supset Q$. Then, for any element $x\in P$ with $x\notin Q$, we have $P\supseteq Q_1=R_PxR_P\supset Q$. Since Q_1 is P-primary by Lemma 6, it must be equal to P. Then $P=R_PxR_P=zR_P=R_Pz$ by Property (D) and Lemma 2, which contradicts to $P=P^2$. Thus $P=Q$.

(3) (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): For $a\in P-A$, we have $A\not\supseteq RaR$, and so $A\subset RaR\subseteq P$ by Property (A). Hence $P=\sqrt{A}\subseteq\sqrt{RaR}\subseteq P$, and so $P=\sqrt{RaR}$.

(iii) $\Rightarrow$ (iv): Assume that $P=\sqrt{RaR}$. For any prime ideal P' such as $P'\subset P$, we have $a\notin P'$, and so $a\in P-\cup\{P': \text{prime ideal} \mid P'\subset P\}$.

(iv) $\Rightarrow$ (v): By Lemma 11, $\cup\{P': \text{prime ideal} \mid P'\subset P\}$ is a prime ideal. So we may take this prime ideal as M.

(v) $\Rightarrow$ (i): If P is not idempotent, then by Corollary 7, P^2 is a P-primary ideal which is different from P. In the case P is idempotent, let $x\in P-M$ and put $Q=RxR$. Then we have $Q_P\subseteq P$, because $P=J(R_P)$. By Property (D), there is a $z\in R_P$ such that $Q_P=R_PxR_P=zT=Tz$, where $T=O_r(R_PxR_P)$, and so Q_P is not idempotent, hence $Q_P\subset P$. Further, since $Q\not\subseteq M$, we have $M\subset Q_P\subset P$ by Property (A), and hence $\sqrt{Q_P}=P$. Thus Q_P is a P-primary ideal of R which is different from P by Lemma 6.

(4) follows immediately from (3).

Corollary 13. *Let P be a prime ideal R and let $p = P \cap V$. Then*
(1) *p is branched if and only if P is branched.*
(2) *p is idempotent if and only if P is idempotent. In this case, we have $P = pR$.*

Proof. (1) Assume that p is unbranched. Then $p = \cup \{p_\lambda \mid p_\lambda (\subset p):$ prime ideal$\}$ by Theorem 12 (4). By Theorem 1 (2) of [3, §3], there is a prime ideal P_λ of R such that $P_\lambda \cap V = p_\lambda$ and $P_\lambda \subset P$. Then $\cup P_\lambda$ is a prime ideal of R by Lemma 11, and $(\cup P_\lambda) \cap V = \cup (P_\lambda \cap V) = \cup p_\lambda$. Hence $\cup P_\lambda = P$ by Theorem 1 (2) of [3, §2], and so P is unbranched by Theorem 12 (4). The converse is proved similarly.
(2) If p is idempotent, then pR is an idempotent ideal of R, and so pR is a prime ideal of R by Lemma 10 (1). Further, since $p = P \cap V \supseteq pR \cap V \supseteq p$, we have $P = pR$ by Theorem 1 (2) of [3, §2], and hence P is idempotent. Conversely assume that P is idempotent. Then, by Theorem 1 (6) of [3, §2], we have $P = pR$, and so $pR = P = P^2 = p^2R$. Hence we have $p = p^2$ by following Lemma 14.

Lemma 14. *Let $\mathscr{A}$ and $\mathscr{B}$ be ideals of V. If $\mathscr{A}R = \mathscr{B}R$, then $\mathscr{A} = \mathscr{B}$.*

Proof. Let $a\ (\neq 0) \in \mathscr{A}$. Then $a = \sum_{i=1}^{n} b_i r_i$, where $b_i \in \mathscr{B}$, $r_i \in R$. Since V is a valuation domain, we have $b_1V + \cdots + b_nV = bV$ for some $b \in \mathscr{B}$. Let $b_i = bv_i$, where $v_i \in V$. Then $a = \sum_{i=1}^{n} b_i r_i = \sum_{i=1}^{n} bv_i r_i = b(\sum_{i=1}^{n} v_i r_i)$, and so $b^{-1}a = \sum_{i=1}^{n} v_i r_i \in K \cap R = V$. Hence $a = b(\sum_{i=1}^{n} v_i r_i) \in bV \subseteq \mathscr{B}$, and so we have $\mathscr{A} \subseteq \mathscr{B}$. The converse inclusion is proved similarly.

Finally, we give an example of a non-commutative valuation ring R such that there exists some prime ideal P of R with $P \supset pR$, where $p = P \cap V$.

Example 15 (see Lemma 1.3 of [4]). Let $V = Z_2$, the localization of the ring of integers Z with respect to $2Z$ and let $K = Q$, the field of rational numbers. Let $\Sigma = D_\tau = K \oplus Ki \oplus Kj \oplus Kij$, where $i^2 = -1$, $j^2 = \tau$, $ij = -ji$ and $\tau = p_1 \cdots p_t$, $p_1, \cdots, p_t$ being distinct primes $\equiv 3 \pmod 4$. In the case $\tau \equiv -1 \pmod 4$, $R = V \oplus Vi \oplus Vj \oplus Vt$ where $t = (1 + i + j + ij)/2$, is a maximal order with $J(R) = (1 + i)R$, and $R/J(R)$ is a division ring by Lemma 1.3 of [4]. Further we have $J(R)^2 = 2R$ and $J(R) \cap V = 2V$, and so $J(R) \supset J(R)^2 = (J(R) \cap V)R$. On the other hand, by Corollary to Proposition 3.3 of [1], R is a local Dedekind ring, and so R is a non-commutative valuation ring.

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