# 21. *Some Problems of Diophantine Approximation in the Theory of the Riemann Zeta Function. II*

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The purpose of the present article is to present some refinements and extensions of our previous work [6]. The details will appear elsewhere.

Let $\alpha$ be a positive number. In [6], we have been concerned with the distribution of

$$\left\{\alpha \frac{\gamma}{2\pi}\right\} - \frac{1}{2}$$

or of

$$\left\{\alpha \frac{\gamma}{2\pi}\right\}^2 - \left\{\alpha \frac{\gamma}{2\pi}\right\} + \frac{1}{6},$$

where $\gamma$ runs over the positive imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$ and $\{x\}$ is the fractional part of $x$. To study this problem we need to use most of the delicate properties of the distribution of the zeros of $\zeta(s)$ as we have already seen in [6]. So to refine our previous results, we need to improve the properties used there or to use the more delicate properties than before. The problem has also another face. It depends heavily on the diophantine properties of $\alpha$. The deep results of Baker [1] have provided us to overcome some of the difficulties as we have also already seen in the previous work [6].

We start with recalling our previous results. Let $N(T)$ denote the number of the zeros of $\zeta(s)$ in $0 < \Im s < T$, which is known to be $\sim \frac{T}{2\pi} \log T$.

Let R. H. be the abbreviation of the Riemann Hypothesis. Then we have proved in [6] the following results.

( I ) *For any positive $\alpha$ and $T > T_0$, we have*

$$\frac{1}{N(T)} \sum_{\gamma \leq T} \left(\left\{\alpha \frac{\gamma}{2\pi}\right\} - \frac{1}{2}\right) \ll \sqrt{\frac{\log\log T}{\log T}}.$$

( II ) (*Under R. H.*) *For any positive $\alpha$, positive $\varepsilon$ and $T > T_0$, we have*

$$\frac{1}{N(T)} \sum_{\gamma \leq T} \left(\left\{\alpha \frac{\gamma}{2\pi}\right\} - \frac{1}{2}\right) \ll \frac{1}{(\log T)^{1-\varepsilon}}.$$

( III ) *For any positive $\alpha$ and $T > T_0$, we have*

$$\frac{1}{N(T)} \sum_{\gamma \leq T} \left(\left\{\alpha \frac{\gamma}{2\pi}\right\}^2 - \left\{\alpha \frac{\gamma}{2\pi}\right\} + \frac{1}{6}\right) \ll \frac{\log\log T}{\log T}.$$

( IV ) (*Under R. H.*) *Suppose that either $\alpha$ or $e^{\alpha}$ is algebraic. Then for any positive $\varepsilon$ and $T > T_0$, we have*

$$\sum_{\gamma \leq T} \left(\left\{\alpha \frac{\gamma}{2\pi}\right\}^2 - \left\{\alpha \frac{\gamma}{2\pi}\right\} + \frac{1}{6}\right)$$

$$= -\frac{T}{2\pi^3}\frac{\Lambda(e^{G\alpha})}{G^2} Li_2\,(e^{-\frac{G}{2}\alpha}) + O\Big(\frac{T}{(\log T)^{1-\varepsilon}}\Big),$$

*where* $\Lambda(x) = \log p$ *if* $x = p^k$ *with a prime number* $p$ *and an integer* $k \geq 1$, $= 0$ *otherwise, we put*

$$Li_2(x) = \sum_{n=1}^{\infty} x^n / n^2$$

*and* $G$ *is either the minimum integer* $n(\geq 1)$ *such that* $e^{n\alpha}$ *is a prime power, or* $1/\alpha$ *if such* $n$ *does not exist.*

(V) (*Under R. H.*) *For any positive* $\alpha$ *and* $T > T_0$, *we have*

$$\frac{1}{N(T)} \sum_{\gamma \leq T} \Big(\Big\{\alpha\frac{\gamma}{2\pi}\Big\}^2 - \Big\{\alpha\frac{\gamma}{2\pi}\Big\} + \frac{1}{6}\Big) \ll \frac{1}{\log T}.$$

We first notice that we can improve all of these results as follows.

(I) can be improved to the following theorem.

**Theorem 1.** *For any positive* $\alpha$ *and* $T > T_0$, *we have*

$$\frac{1}{N(T)} \sum_{\gamma \leq T} \Big(\Big\{\alpha\frac{\gamma}{2\pi}\Big\} - \frac{1}{2}\Big) \ll \Big(\frac{\log\log T}{\log T}\Big)^{\frac{4}{5}}.$$

(IV) can be improved to the following theorem which does not need even any unproved hypothesis.

**Theorem 2.** *Suppose that either* $\alpha$ *or* $e^{\alpha}$ *is algebraic. Then for any* $T > T_0$, *we have*

$$\sum_{\gamma \leq T} \Big(\Big\{\alpha\frac{\gamma}{2\pi}\Big\}^2 - \Big\{\alpha\frac{\gamma}{2\pi}\Big\} + \frac{1}{6}\Big)$$
$$= -\frac{T}{2\pi^3}\frac{\Lambda(e^{G\alpha})}{G^2} Li_2\,(e^{-\frac{G}{2}\alpha}) + O\Big(\frac{T}{\log T}(\log\log T)^2\Big).$$

For a general $\alpha$, we can show the following theorem.

**Theorem 3.** *For any positive* $\alpha$ *and* $T > T_0$, *we have*

$$\sum_{\gamma \leq T} \Big(\Big\{\alpha\frac{\gamma}{2\pi}\Big\}^2 - \Big\{\alpha\frac{\gamma}{2\pi}\Big\} + \frac{1}{6}\Big)$$
$$= -\frac{T}{2\pi^3}\frac{\Lambda(e^{G\alpha})}{G^2} Li_2\,(e^{-\frac{G}{2}\alpha}) + O\Big(\sum_{n<T_1}\frac{\Lambda(n)}{\sqrt{n}\,(\log n)^2}\min\Big(T, \frac{1}{\big\|\frac{1}{\alpha}\log n\big\|}\Big)\Big)$$
$$+ O\Big(\frac{T}{\log T}(\log\log T)^2\Big),$$

*where we put* $T_1 = T^{\frac{a}{\log\log T}}$ *with an arbitrarily small positive* $a$ *and* $\|x\|$ *denotes the distance of* $x$ *from a nearest integer.*

Theorem 3 implies the following result.

**Theorem 4.** *For any positive* $\alpha$ *and* $T > T_0$, *we have*

$$\sum_{\gamma \leq T} \Big(\Big\{\alpha\frac{\gamma}{2\pi}\Big\}^2 - \Big\{\alpha\frac{\gamma}{2\pi}\Big\} + \frac{1}{6}\Big) \ll T.$$

This eliminates the assumption R. H. from (V) and also improves upon (III). Finally, (II) can be improved to the following result.

**Theorem 5** (*Under R. H.*). *For any positive* $\alpha$ *and* $T > T_0$, *we have*

$$\frac{1}{N(T)} \sum_{\gamma \leq T} \Big(\Big\{\alpha\frac{\gamma}{2\pi}\Big\} - \frac{1}{2}\Big) \ll \frac{(\log\log T)^2}{\log T}.$$

To prove (I) — (V), we have used the following (cf. Fujii [3]).

**Lemma.** *For any $\alpha > 0$, any positive $\varepsilon$ and for $T > T_0$, we have*

$$\frac{1}{N(T)}\left|\left\{\gamma \le T\,;\, 0 \le \left\{\alpha\frac{\gamma}{2\pi}\right\} \le \eta\right\}\right| = \eta + O\left(\frac{1}{(\log T)^{1-\varepsilon}}\right)$$

*uniformly for $\eta$ in $0 \le \eta \le 1$.*

We have also used the estimate (cf. Fujii [5]) on the exponential sum over $\gamma$; for $1 < X \ll T^{\frac{8}{7}-\varepsilon}$, $\varepsilon > 0$ and $T > T_0$,

$$\sum_{\gamma \le T} X^{i\gamma} \ll T\log X + \min\left(\frac{\log T}{\log X},\, T\log T\right).$$

Under R. H., the following estimate (cf. Fujii [5]) has been used; for $X > 1$ and $T > T_0$,

$$\sum_{\gamma \le T} X^{i\gamma} = -\frac{T}{2\pi}\frac{\Lambda(X)}{\sqrt{X}} + O\left(\frac{\log T}{\log X} + \sqrt{X}\log X \frac{\log T}{(\log\log T)^2}\right.$$
$$\left. + \sqrt{X}\log(3X)\log\log(3X) + \frac{\log(2X)}{\sqrt{X}}\min\left(T, \frac{1}{\left|\log \frac{X}{P(X)}\right|}\right)\right),$$

where $P(X)$ is the nearest prime power other than $X$ itself.

To prove Theorems 1-4, we use the above Lemma and the mean value theorem on

$$\int_0^T S^{2k}(t)\,dt \quad \text{and} \quad \int_0^T\int_0^y S(t)\,dt\,dy,$$

where we put

$$S(t) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + it\right)$$

as usual.

To prove Theorem 5, we use the following which refines above Lemma.

**Theorem 6.** *For any $\alpha > 0$ and for $T > T_0$, we have*

$$\frac{1}{N(T)}\left|\left\{\gamma \le T\,;\, 0 \le \left\{\alpha\frac{\gamma}{2\pi}\right\} \le \eta\right\}\right| = \eta + O\left(\frac{\log\log T}{\log T}\right)$$

*uniformly for $\eta$ in $0 \le \eta \le 1$.*

This is a special case of the following theorem which gives a refinement of Theorem in Fujii [3].

**Theorem 6'.** *Suppose that $T > T_0$ and $\frac{1}{\log T} \le h \ll \sqrt{\frac{T\log\log T}{\log T}}$. Then we have*

$$\sup_{0\le\eta\le1}\left|\frac{1}{N(T)}\left|\left\{\gamma \le T\,;\, 0 \le \left\{\frac{\gamma}{h}\right\} \le \eta\right\}\right| - \eta\right| \ll \frac{\log(h\log T + 3)}{h\log T}.$$

To get this theorem, we use the following theorem which is a refinement of Main Theorem of Fujii [2] and can be proved in a similar manner.

**Theorem 7.** *Suppose that $T > T_0$, $0 < h \ll T$ and $1 \le k \ll \frac{\log T}{\log\log T}$. Then we have*

$$\int_0^T (S(t+h) - S(t))^{2k}\,dt$$
$$= \frac{2k!}{(2\pi)^{2k}k!}2^k T\left(\mathrm{Cin}\left(h\log\frac{T}{2\pi}\right) - \mathrm{Cin}(h\log 2)\right)^k$$

$$+ O\Big(T(Ak)^k\Big(\Big(\mathrm{Cin}\Big(h\log\frac{T}{2\pi}\Big) - \mathrm{Cin}(h\log 2)\Big)^{k-\frac{1}{2}}$$
$$+ \Big(\mathrm{Cin}\Big(h\log\frac{T}{2\pi}\Big) - \mathrm{Cin}(h\log 2)\Big)^{k-1}\log\log(h+3) + k^k$$
$$+ (\log\log(h+3))^k\Big)\Big),$$

*where $A$ is some positive absolute constant and* $\mathrm{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\,dt$.

We notice that

$$\mathrm{Cin}\Big(h\log\frac{T}{2\pi}\Big) - \mathrm{Cin}(h\log 2)$$
$$= \begin{cases} \mathrm{Cin}\Big(h\log\frac{T}{2\pi}\Big) + O\Big(\frac{1}{\log^2 T}\Big) & \text{if } 0 < h \ll \frac{1}{\log T} \\ \log\Big(h\log\frac{T}{2\pi}\Big) + \int_0^1 \frac{1-\cos t}{t}\,dt & \\ \qquad - \mathrm{Cin}(h\log 2) + O\Big(\frac{1}{h\log T}\Big) & \text{if } \frac{1}{\log T} \ll h \ll 1 \\ \log\log T + O(1) & \text{if } 1 \ll h. \end{cases}$$

We may recall here that for $k=1$, the following mean value theorem has been proved in Fujii [4] under the Riemann Hypothesis and Montgomery's conjecture ;for $T > T_0$ and for $0 < \alpha = o(\log T)$,

$$\int_0^T \Big(S\Big(t + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\Big) - S(t)\Big)^2 dt$$
$$= \frac{T}{\pi^2}(\mathrm{Cin}(2\pi\alpha) + 1 - \cos(2\pi\alpha) + \pi^2\alpha - 2\pi\alpha\mathrm{Si}(2\pi\alpha)) + o(T).$$

Before turning to the next topic we mention the following two remarks.

**Remark 1.** In Theorem 2, if we assume only that $e^\alpha$ is algebraic, then the remainder term $\Big(\frac{T}{\log T}(\log\log T)^2\Big)$ can be replaced by $O\Big(\frac{T}{\log T}\sqrt{\log\log T}\Big)$.

**Remark 2.** If we assume R. H., then each remainder term

$$O\Big(\frac{T(\log\log T)^2}{\log T}\Big)$$

in Theorems 2 and 3 can be replaced by

$$O\Big(\frac{T}{\log T}\log\log T\Big).$$

Moreover, if $e^\alpha$ is algebraic, then it is replaced by $O\Big(\frac{T}{\log T}\Big)$.

Now we turn our attentions to the extension of our problem. We extend our porblem to the distribution of

$$\{\alpha\gamma^\theta\} - \frac{1}{2}$$

and

$$\{\alpha\gamma^\theta\}^2 - \{\alpha\gamma^\theta\} + \frac{1}{6},$$

where $\theta$ is a positive number $<1$ and $\alpha$ is a positive number. This distribution represents the properties quite different from the previous one.

In fact, in this case, as we shall see below, the clear dependence on such nature as $\alpha$ is related with the prime numbers disappears at least in the first main term of the asymptotic formula. We do not also use Baker's theorem at least to get the results which will be described below.

To describe our results, we denote the $n$-th Bernoulli polynomial by $B_n(x)$. Namely, $B_1(x)=x-\frac{1}{2}$, $B_2(x)=x^2-x+\frac{1}{6}$, $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$ and $B_4(x)=x^4-2x^3+x^2-\frac{1}{30}$, for example. We can prove the following theorems.

**Theorem 8.** *Let $\theta$ be a positive number $<1$ and $\alpha$ be a positive number. Then we have for $T>T_0$ and for any small positive $\varepsilon$,*

$$\sum_{\gamma\leq T}\left(\{\alpha\gamma^\theta\}-\frac{1}{2}\right)=\frac{1}{4\pi\alpha\theta}B_2(\{\alpha T^\theta\})T^{1-\theta}\log\frac{T}{2\pi}+F_1(\alpha,\theta,T)$$
$$+O(T^\theta\log^2 T)+O(T^{\frac{1}{5}+\frac{4\theta}{5}}(\log T)^{\frac{4}{5}+\varepsilon})$$

*where we put*

$$F_1(\alpha,\theta,T)\equiv-\frac{1}{4\pi^3\theta\alpha^{\frac{1}{\theta}}}\int_1^{\alpha T^\theta}\left(\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^2}\right)x^{\frac{1}{\theta}-2}\left(\frac{1}{\theta}+\left(\frac{1}{\theta}-1\right)\log\frac{x^{\frac{1}{\theta}}}{2\pi\alpha^{\frac{1}{\theta}}}\right)dx$$
$$=-\frac{1-\theta}{12\pi\alpha^2\theta}B_3(\{\alpha T^\theta\})T^{1-2\theta}\log T$$
$$-\frac{1-(1-\theta)\left(\left(1+\frac{1}{\theta}\right)\log(2\pi\alpha)+\log(2\pi)\right)}{12\pi\alpha^2\theta^2}B_3(\{\alpha T^\theta\})T^{1-2\theta}$$
$$+O(T^{1-3\theta}\log T).$$

This implies, in particular, the following.

**Corollary 1.** *Suppose that $0<\theta\leq 4/9$. Then for any positive $\alpha$ and $T>T_0$, we have*

$$\sum_{\gamma\leq T}\left(\{\alpha\,\gamma^\theta\}-\frac{1}{2}\right)\sim\frac{1}{4\pi\alpha\theta}B_2(\{\alpha T^\theta\})T^{1-\theta}\log\frac{T}{2\pi}.$$

**Theorem 9.** *Let $\theta$ be a positive number $<1$ and $\alpha$ be a positive number. Then we have for $T>T_0$*

$$\sum_{\gamma\leq T}\left(\{\alpha\,\gamma^\theta\}^2-\{\alpha\gamma^\theta\}+\frac{1}{6}\right)=\frac{1}{6\pi\alpha\theta}B_3(\{\alpha T^\theta\})T^{1-\theta}\log\frac{T}{2\pi}$$
$$+F_2(\alpha,\theta,T)+O(\Psi(\theta,T))+O(\log T),$$

*where we put*

$$\Psi(\theta,T)=\begin{cases}T^{2\theta-1}\log T & \text{for } 1/2<\theta<1\\ \log^2 T & \text{for } \theta=1/2\\ 1 & \text{for } 0<\theta<1/2\end{cases}$$

*and*

$$F_2(\alpha,\theta,T)\equiv-\frac{1}{4\pi^4\theta\alpha^{\frac{1}{\theta}}}\int_1^{\alpha T^\theta}\left(\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^3}\right)x^{\frac{1}{\theta}-2}$$

$$\left(\frac{1}{\theta} + \left(\frac{1}{\theta} - 1\right) \log \frac{x^{\frac{1}{\theta}}}{2\pi\alpha^{\frac{1}{\theta}}}\right) dx$$

$$= \frac{1-\theta}{24\pi\alpha^2\theta} B_4(\{\alpha T^{\theta}\}) T^{1-2\theta} \log T$$

$$- \frac{1 - (1-\theta)\left(\left(1 + \frac{1}{\theta}\right) \log(2\pi\alpha) + \log(2\pi)\right)}{8\pi^5\alpha^2\theta^2} B_4(\{\alpha T^{\theta}\}) T^{1-2\theta}$$

$$+ O(T^{1-3\theta} \log T).$$

This implies, in particular, the following.

**Corollary 2.** *Suppose that* $0 < \theta < 2/3$. *Then for any positive* $\alpha$, *we have*

$$\sum_{\gamma \le T} \left(\{\alpha\gamma^{\theta}\}^2 - \{\alpha\gamma^{\theta}\} + \frac{1}{6}\right) \sim \frac{1}{6\pi\alpha\theta} B_3(\{\alpha T^{\theta}\}) T^{1-\theta} \log \frac{T}{2\pi}.$$

To prove Theorems 8 and 9, we need the following theorem which corresponds to the above Lemma and Theorem 6 for the sequence $\{\alpha(\gamma/2\pi)\}$.

**Theorem 10.** *For any* $\alpha > 0$, *any* $0 < \theta < 1$, *any* $0 \le \kappa < \eta \le 1$ *and for* $T > T_0$, *we have*

$$\frac{1}{N(T)} |\{\gamma \le T\,;\, \kappa \le \{\alpha\gamma^{\theta}\} \le \eta\}| = (\eta - \kappa) + O\left((\eta - \kappa)\frac{1}{T^{\theta}}\right) + O\left(\frac{1}{T^{1-\theta}}\right).$$

If we assume R. H., then we can refine Theorem 9 slightly as follows.

**Theorem 11** (Under R. H.). *Let* $\theta$ *be a positive number* $< 1$ *and* $\alpha$ *be a positive number. Then we have for* $T > T_0$

$$\sum_{\gamma \le T} \left(\{\alpha\gamma^{\theta}\}^2 - \{\alpha\gamma^{\theta}\} + \frac{1}{6}\right) = \frac{1}{6\pi\alpha\theta} B_3(\{\alpha T^{\theta}\}) T^{1-\theta} \log \frac{T}{2\pi}$$

$$+ F_2(\alpha, \theta, T) + O(\tilde{\Psi}(\theta, T)) + O\left(\frac{\log T}{\log\log T}\right),$$

*where we put*

$$\tilde{\Psi}(\theta, T) = \begin{cases} \dfrac{T^{2\theta-1}}{\log T} & \text{for } 1/2 < \theta < 1 \\ \left(\dfrac{\log T}{\log\log T}\right)^2 & \text{for } \theta = 1/2 \\ 1 & \text{for } 0 < \theta < 1/2. \end{cases}$$

Finally, we remark that we can also treat the sum

$$B_n(\{\alpha\gamma/2\pi\}) \quad \text{or} \quad B_n(\{\alpha\gamma^{\theta}\}) \quad \text{for} \quad n \ge 3.$$

## References

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