30. *A Determinant Formula for Period Integrals*

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We prove a formula for the determinant of the period integrals. It is expressed as the product of the pairing with the relative canonical cycle and special values of Γ-function. It generalizes a previous result for $\boldsymbol{P}^1$ [3]. It can be regarded as a Hodge analogue of the formula for l-adic cohomology [2]. By combining with this, it proves a part of a conjecture of Deligne: A motive of rank 1 over a number field is defined by an algebraic Hecke character ([1] Conjecture 8.1 (iii)), in a certain special case.

1. Definition of the period. Let k, F be subfields of the complex number field $\boldsymbol{C}$ and U be a smooth separated scheme over k of dimension n. We consider the category $M_k(U, F)$ consisting of triples $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$ as follows

(1) A locally free $\mathcal{O}_U$-module $\mathcal{E}$ of finite rank with an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_U^1$ which is regular singular along the boundary.

(2) A local system V of F-vector spaces on the complex manifold U^{an}.

(3) An isomorphism $\rho : V \otimes_F \boldsymbol{C} \xrightarrow{\sim} Ker \nabla^{an}$ of local systems of $\boldsymbol{C}$-vector spaces on U^{an}.

We explain the terminology. Let X be a proper smooth scheme over k containing U as a dense open subscheme such that the complement $D = X - U$ is a divisor with simple normal crossings. A divisor is said to have simple normal crossings if its irreducible components D_i are smooth and their $m \times m$ intersections are transversal. An integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_U^1$ is said to be regular singular along the boundary if there exists a locally free $\mathcal{O}_X$-module $\mathcal{E}_X$ and a logarithmic integrable connection $\nabla_X : \mathcal{E}_X \to \mathcal{E}_X \otimes \Omega_X^1(\log D)$ extending $(\mathcal{E}, \nabla)$. It is independent of the choice of compactification X. The complex manifold of the $\boldsymbol{C}$-valued points of U is denoted by U^{an} and the algebraic connection ∇ induces an analytic connection ∇^{an} on U^{an}

We define the determinant of the period

$$per(\mathcal{M}) \in k^\times \backslash \boldsymbol{C}^\times / F^\times$$

for an object $\mathcal{M} \in M_k(U, F)$. Let $MPic_k(U, F)$ be the group of isomorphism class of the objects of $M_k(U, F)$ of rank 1 with respect to the tensor product. For $U = \operatorname{Spec} k$, we identify $MPic_k(\operatorname{Spec} k, F)$ with $k^\times \backslash \boldsymbol{C}^\times / F^\times$ by $[\mathcal{M}] \to \rho(v)/e$ for $\mathcal{M} \in M_k(\operatorname{Spec} k, F)$ of rank 1 with basis $e \in \mathcal{E}$ and $v \in V$. For $\mathcal{M} \in M_k(U, L)$, we define $per(\mathcal{M}) \in k^\times \backslash \boldsymbol{C}^\times / F^\times$ as $[\det R\Gamma(U, \mathcal{M})] \in MPic_k(\operatorname{Spec} k, F)$ defined below. Let $DR(\mathcal{E})$ be the de Rham complex

$$[\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_U^1 \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_U^2 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_U^n].$$

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Since $H^q(U, DR(\mathscr{E})) \otimes_k \boldsymbol{C} \simeq H^q(U^{an}, DR(\mathscr{E})^{an})$ by GAGA, the isomorphism ρ induces $H^q(\rho) : H^q(U, DR(\mathscr{E})) \otimes_k \boldsymbol{C} \simeq H^q(U^{an}, V) \otimes_F \boldsymbol{C}$. In other words, the triple

$$H^q(U, \mathscr{M}) = (H^q(U, DR(\mathscr{E})), H^q(U^{an}, V), H^q(\rho))$$

is an object of $M_k(\mathrm{Spec}\ k, F)$. Taking the alternating tensor product of the determinant, we obtain

$$\det R\Gamma(U, \mathscr{M}) = (\otimes_q \det H^q(U, DR(\mathscr{E}))^{\otimes(-1)^q}, \otimes_q \det H^q(U^{an}, V)^{\otimes(-1)^q}, \otimes_q \det H^q(\rho)^{\otimes(-1)^q}).$$

The period $per(\mathscr{M}) \in k^\times \backslash \boldsymbol{C}^\times / L^\times$ is thus defined.

2. The relative Chow group. In this section, we define the relative Chow group $CH^n(X \bmod D)$ of dimension 0 and the relative canonical cycle $c_{X \bmod D} \in CH^n(X \bmod D)$. They are slight modifications of those in [2]. Let X be a smooth scheme over a field k of dimension n and $D = \cup_{i \in I} D_i$ be a divisor with simple normal crossings. Let $\mathscr{K}_n(X)$ denote the sheaf of Quillen's K-group on X_{Zar}. Namely the Zariski sheafification of the presheaf $U \to K_n(U)$. Let $\mathscr{K}_n(X \bmod D)$ be the complex $[\mathscr{K}_n(X) \to \oplus_i \mathscr{K}_n(D_i)]$. Here $\mathscr{K}_n(X)$ is put on degree 0 and $\mathscr{K}_n(D_i)$ denotes their direct image on X. It is the truncation at degree 1 of the complex $\mathscr{K}_{n,X,D}$ studied in [2] and there is a natural map $\mathscr{K}_{n,X,D} \to \mathscr{K}_n(X \bmod D)$. We call the hypercohomology $H^n(X, \mathscr{K}_n(X \bmod D))$ the relative Chow group of dimension 0 and write

$$CH^n(X \bmod D) = H^n(X, \mathscr{K}_n(X \bmod D)).$$

We recall the definition of the relative canonical class

$$c_{X \bmod D} = (-1)^n c_n(\Omega^1_X(\log D), res) \in CH^n(X \bmod D).$$

Let V be the covariant vector bundle associated to the locally free $\mathscr{O}_X$-module $\Omega^1_X(\log D)$ of rank n. For each irreducible component D_i, let $\Delta_i = r_i^{-1}(1)$, where $r_i : V|_{D_i} \to \boldsymbol{A}^1_{D_i}$ is induced by the Poincaré residue $res_i : \Omega^1_X(\log D)|_{D_i} \to \mathscr{O}_{D_i}$ and $1 \subset \boldsymbol{A}^1$ is the 1-section. Let $\mathscr{K}_n(V \bmod \Delta)$ be the complex $[\mathscr{K}_n(V) \to \oplus_i \mathscr{K}_n(\Delta_i)]$ defined similarly as above and $\{0\} \subset V$ be the zero section. Then we have

$$\begin{array}{l} H^n_{\{0\}}(V, \mathscr{K}_n(V \bmod \Delta)) \simeq H^n_{\{0\}}(V, \mathscr{K}_n(V)) \simeq H^0(X, \boldsymbol{Z}) \\ \quad\downarrow \\ H^n(V, \mathscr{K}_n(V \bmod \Delta)) \simeq H^n(X, \mathscr{K}_n(X \bmod D)) = CH^n(X \bmod D) \end{array}$$

by the purity and homotopy property of K-cohomology. The relative top chern class $c_n(\Omega^1_X(\log D), res) \in CH^n(X \bmod D)$ is defined as the image of $1 \in H^0(X, \boldsymbol{Z})$.

In the rest of this section, we give an adelic presentation

$$CH^n(X \bmod D) \simeq Coker(\partial : \bigoplus_{y \in X_1} H^{n-1}_y \to \bigoplus_{x \in X_0} H^n_x).$$

Here X_i denotes the set of the points of X of dimension i and the groups H^{n-1}_y and H^n_x and the homomorphism ∂ are defined as follows.

(1) The group H^n_x for $x \in X_0$. It is an extension of $\boldsymbol{Z}$ by $\oplus_{i \in I_x} \kappa(x)^\times$ with the index set $I_x = \{i ; x \in D_i\}$. For $i \in I_x$, let $N_i(x)$ be the one-dimensional $\kappa(x)$-vector space $\mathscr{O}_X(-D_i) \otimes \kappa(x)$. The $\kappa(x)$-algebra $\oplus_{m \in \boldsymbol{Z}} N_i(x)^{\otimes m}$ is non-canonically isomorphic to the Laurent polynomial ring $\kappa(x)[T, T^{-1}]$. We put $H^n_{x,i} = (\otimes_{m \in \boldsymbol{Z}} N_i(x)^{\otimes m})^\times$. It is an extension of $\boldsymbol{Z}$ by

$\kappa(x)^\times$. By pulling-back $\oplus_{i\in I_x} H^n_{x,i}$ by the diagonal $\boldsymbol{Z}\to\oplus_i \boldsymbol{Z}$, we obtain H^n_x by

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{i\in I_x}\kappa(x)^\times \arrow[r] \arrow[d, equal] & H^n_x \arrow[r] \arrow[d] & \boldsymbol{Z} \arrow[r] \arrow[d, "\text{diagonal}"] & 0 \\ 0 \arrow[r] & \bigoplus_{i\in I_x}\kappa(x)^\times \arrow[r] & \bigoplus_{i\in I_x} H^n_{x,i} \arrow[r] & \bigoplus_{i\in I_x}\boldsymbol{Z} \arrow[r] & 0. \end{tikzcd}$$

(2) The group H^{n-1}_y for $y\in X_1$. It is an extension of $\kappa(y)^\times$ by $\oplus_{i\in I_y} K_2(\kappa(y))$ with the index set $I_y=\{i\ ;\ y\in D_i\}$. In the same way as above, we define an extension H'_y(resp. $H'_{y,i}$) of $\boldsymbol{Z}$ by $\oplus_{i\in I_y}\kappa(y)^\times$ (resp. by $\kappa(y)^\times$ for $i\in I_y$). The tensor product $H'_y\otimes\kappa(y)^\times$ is an extension of $\kappa(y)^\times$ by $\otimes_{i\in I_y}(\kappa(y)^\times\otimes\kappa(y)^\times)$. By pushing it by the symbol map $\kappa(y)^\times\otimes\kappa(y)^\times\to K_2(\kappa(y))$ we obtain H^{n-1}_y by

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{i\in I_y}(\kappa(y)^\times\otimes\kappa(y)^\times) \arrow[r] \arrow[d, "\text{symbol}"'] & H'_y\otimes\kappa(y)^\times \arrow[r] \arrow[d] & \kappa(y)^\times \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \bigoplus_{i\in I_y} K_2(\kappa(y)) \arrow[r] & H^{n-1}_y \arrow[r] & \kappa(y)^\times \arrow[r] & 0. \end{tikzcd}$$

(3) The homomorphism ∂. It is the direct sum of the (x, y)-component $\partial_{x,y}: H^{n-1}_y\to H^n_x$ for $x\in X_0$ and $y\in X_1$. This fits in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{i\in I_y} K_2(\kappa(y)^\times) \arrow[r] \arrow[d, "\bigoplus_{i\in I_y}(\ ,\ )_x"'] & H^{n-1}_y \arrow[r] \arrow[d, "\partial_{x,y}"] & \kappa(y)^\times \arrow[r] \arrow[d, "\mathrm{ord}_x"] & 0 \\ 0 \arrow[r] & \bigoplus_{i\in I_x}\kappa(x)^\times \arrow[r] & H^n_x \arrow[r] & \boldsymbol{Z} \arrow[r] & 0 \end{tikzcd}$$

and is 0 unless x is not in the closure Y of $\{y\}$. Here $\mathrm{ord}_x:\kappa(y)^\times\to\boldsymbol{Z}$ is the usual order and $(\ ,\ )_x: K_2(\kappa(y))\to\kappa(x)^\times$ is the tame symbol. If $\{\tilde{x}_j\}_j$ denote the inverse image of x in the normalization of Y, they are defined by $\mathrm{ord}_x(f)=\sum_j[\kappa(\tilde{x}_j):\kappa(x)]\cdot\mathrm{ord}_{\tilde{x}}(f)$ and $(f, g)_x=\Pi_j N_{\kappa(\tilde{x}_j)/\kappa(x)}(f, g)_{\tilde{x}_j}$ for $f, g\in\kappa(y)^\times$. Here $\mathrm{ord}_{\tilde{x}_j}$ is the valuation, $(f, g)_{\tilde{x}_j}=((-1)^{\mathrm{ord}_{\tilde{x}_j}(f)\mathrm{ord}_{\tilde{x}_j}(g)} f^{\mathrm{ord}_{\tilde{x}_j}(g)} g^{-\mathrm{ord}_{\tilde{x}_j}(f)})(\tilde{x}_j)$ is the usual tame symbol and N denotes the norm.

To give the definition of $\partial_{x,y}$, we introduce the tame symbol for invertible sheaves. For an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ and $f\in\kappa(y)^\times$, let $(\mathcal{L}, f)_x$ be the one-dimensional $\kappa(x)$-vector space generated by the symbol $(l, f)_x$ for a non-zero rational section l of $\mathcal{L}$. We put $(gl, f)_x=(g, f)_x(l, f)_x$ for other section gl of $\mathcal{L}$ and $g\in\kappa(y)^\times$. We have a canonical isomorphism $(\mathcal{L}, f)_x\simeq\mathcal{L}(x)^{\otimes\mathrm{ord}_x(f)}$ where $\mathcal{L}(x)=\mathcal{L}\otimes\kappa(x)$. In fact, we may assume $\mathcal{O}_{Y,x}$ is normal by considering the norm and then $(l, f)_x\mapsto((-1)^{\mathrm{ord}_x(l)\mathrm{ord}_x(f)} l^{\otimes\mathrm{ord}_x(f)} f^{-\mathrm{ord}_x(l)})(x)$ gives the isomorphism.

Let x be a closed point in the closure Y of $\{y\}$ and D_i be an irreducible component of D containing x. We define $\partial_{x,y,i}: H'_{y,i}\otimes\kappa(y)^\times\to H^n_{x,i}$ when $y\in D_i$ and $\partial_{x,y,i}:\kappa(y)^\times\to H^n_{x,i}$ otherwise. They induce $\partial_{x,y}$. Let N_i be the invertible $\mathcal{O}_Y$-module $\mathcal{O}_Y(-D_i)$. Note that $H'_{y,i}$ is identified with $\amalg_{m\in\boldsymbol{Z}}(N_i(y)^{\otimes m}-\{0\})$ as a set for $N_i(y)=N_i\otimes\kappa(y)$ and similarly $H^n_{x,i}=\amalg_{m\in\boldsymbol{Z}}(N_i(x)^{\otimes m}-\{0\})$. First assume $y\in D_i$. For $\nu\in N_i(y)^{\otimes m}$, $(\nu\neq 0)$ and $f\in\kappa(y)^\times$, let $(\nu, f)_x\in N_i(x)^{\otimes m\times\mathrm{ord}_x(f)}$, $\neq 0$ be the tame symbol for

$N_i^{\otimes m}$ defined above. Then the map $H'_{y,i} \times \kappa(y)^\times \to H^n_{x,i} : (\nu, f) \mapsto (\nu, f)_x$ induces the map $\partial_{x,y,i} : H'_{y,i} \otimes \kappa(y)^\times \to H^n_{x,i}$. Next we assume $y \notin D_i$. Let $N_i(y)$ be as above and consider $1 \in N_i(y)$. Then $\partial_{x,y,i} : \kappa(y)^\times \to H^n_{x,i}$ is defined by $f \mapsto (1, f)_x$.

The isomorphism $Coker(\partial : \oplus_{y \in X_1} H^{n-1}_y \to \oplus_{x \in X_0} H^n_x) \simeq CH^n(X \bmod D)$ is defined as follows. For the cohomology with support, we have isomorphisms $H^n_x \simeq H^n_x(X, \mathcal{K}_n(X \bmod D))$ for $x \in X_0$ and $H^{n-1}_y \simeq H^{n-1}_y(X, \mathcal{K}_n(X \bmod D))$ for $y \in X_1$. The spectral sequence $E_1^{p,q} = \oplus_{x \in X_{-p}} H^{p+q}_x(X, \mathcal{K}_n(X \bmod D)) \Rightarrow H^{p+q}(X, \mathcal{K}_n(X \bmod D))$ degenerates at E_2-terms and induces the morphism.

3. The pairing. Let k, $F \subset \boldsymbol{C}$ and $X \supset U$ over k be as in section 1. Recall that $MPic_k(U, F)$ denotes the class group of the rank 1 objects of $M_k(U, F)$. In this section, we define a pairing:

$$(\ , \ ) : MPic_k(U, F) \otimes CH^n(X \bmod D) \to MPic_k(\mathrm{Spec}\, k, F) \simeq k^\times \backslash \boldsymbol{C}^\times / F^\times.$$

First we define the local pairing. Let $x \in X_0$ be a closed point of X. Let $\{\bar{x}_j \, ; j \in J_x\}$ be the set of $\boldsymbol{C}$-valued points of X supported on x. For each $\bar{x}_j$, let $\sigma_j : \kappa(x) \to \boldsymbol{C}$ be the corresponding k-morphism. We have an isomorphism $(\sigma_j)_j : \kappa(x) \otimes_k \boldsymbol{C} \to \Pi_{j \in J_x} \boldsymbol{C}$. We define the local pairing

$$(\ , \ )_x : MPic_k(U, F) \otimes H^n_x \to MPic_k(x, F) \simeq (\kappa(x) \otimes 1)^\times \backslash (\kappa(x) \otimes_k \boldsymbol{C})^\times / \prod_j F^\times$$

for $x \in X_0$. When $x \in U$, the pairing $(\ \ , 1)_x$ with $1 \in \boldsymbol{Z} = H^n_x$ is simply defined by taking the fiber at x. We consider the general case.

Let $((\mathcal{E}, \nabla), V, \rho)$ be an object of $M_k(U, F)$ of rank 1. Take an invertible $\mathcal{O}_X$-module $\mathcal{E}_X$ extending $\mathcal{E}$ and an $\mathcal{O}_{X,x}$-basis e of $\mathcal{E}_{X,x}$. Let $I_x = \{i : x \in D_i\}$. For each irreducible component $D_i \ni x$ of D, put $\nabla_i(x) = res_i(\nabla e / e)(x) \in \kappa(x)$. For each $\bar{x}_j$, $j \in J_x$, let Δ_j be a small polydisc in X^{an} with center $\bar{x}_j$ and $\tilde{\Delta}^*_j$ be the universal covering of $\Delta^*_j = \Delta_j \cap U^{an}$. Take a basis v_j of the one-dimensional F-vector space $\Gamma(\tilde{\Delta}^*_j, V)$. This space has a natural action of $\pi_1(\Delta^*_j)$ and the action of the monodromy γ_{ij} around the inverse image of D_i is given by $\exp(-2\pi\sqrt{-1}\,\sigma_j(\nabla_i(x))) \in F^\times$. Let ψ_j be the analytic function $\rho(v_j)/e$ on $\tilde{\Delta}^*_j$.

Let $f \in H^n_x$. The group $H^n_{x,i}$ in the last section is canonically identified with the group $\Gamma(\mathrm{Spec}\, \mathcal{O}_{X,x} - D_i, \mathcal{O}^\times)/(1 + m_x)$ for $i \in I_x$. We take $\varphi_i \in \Gamma(\mathrm{Spec}\, \mathcal{O}_{X,x} - D_i, \mathcal{O}^\times)$ for each $i \in I_x$ representing f by this identification. Let φ_{ij} be the pull-back of φ_i to Δ^*_j and define an analytic function $\varphi_{ij}^{\nabla_i(x)} = \exp(\sigma_j(\nabla_i(x)) \log \varphi_{ij})$ on $\tilde{\Delta}^*_j$. It is well-defined modulo $F^\times$ since the change of the branch of the logarithm multiplies an integral power of $\exp(2\pi\sqrt{-1}\,\sigma_j(\nabla_i(x))) \in F^\times$. We consider an analytic function on $\tilde{\Delta}^*_j$

$$(\psi, \varphi)_j = (-1)^{\mathrm{ord}_x f \Sigma_i \sigma_j(\nabla_i(x))} \cdot \psi_j^{\mathrm{ord}_x f} \prod_{i \in I_x} \varphi_{ij}^{\nabla_i(x)}.$$

Here $\mathrm{ord}_x : H^n_x \to \boldsymbol{Z}$ is the canonical map and $(-1)^{\mathrm{ord}_x f \Sigma_i \sigma_j(\nabla_i(x))} = \exp(\pi\sqrt{-1} \cdot \mathrm{ord}_x f \sum_i \sigma_j(\nabla_i(x)))$. It is a pull-back of an invertible holomorphic function on Δ_j also denoted $(\psi, \varphi)_j$. In fact $\log(\psi, \varphi)_j$ is invariant by monodromy and $d\log(\psi, \varphi)_j$ is holomorphic on Δ_j. Therefore the value $(\psi, \varphi)_j(\bar{x}_j) \in \boldsymbol{C}^\times$ is well-defined modulo $F^\times$. The local pairing is defined by

$$([\mathcal{M}], f)_x = ((\psi,\varphi)_j(\bar{x}_j)) \in (\kappa(x) \otimes 1)^\times \backslash \prod_j \boldsymbol{C}^\times / \prod_j F^\times.$$

The norm $N_{\kappa(x)/k}$ induces $MPic_k(x, F) \to MPic_k(\mathrm{Spec}\, k, F)$ and the local pairings define the global pairing $MPic_k(U, F) \times CH^n(X \bmod D) \to MPic_k(\mathrm{Spec}\, k, F)$. The required reciprocity law follows from that for the tame symbols on a curve and the fact that the residue ∇_i of the connection is constant on each component D_i. For an object $\mathcal{M} \in M_k(U, F)$, the pairing with the relative canonical class defined in the last section

$$(\det \mathcal{M}, c_{X\bmod D}) \in k^\times \backslash \boldsymbol{C}^\times / F^\times$$

is thus defined.

4. Main theorem. Let $k, F \subset \boldsymbol{C}$ and $X \supset U$ over k be as in the previous sections. First we review the residues of a logarithmic integrable connection $\nabla : \mathcal{E}_X \to \mathcal{E}_X \otimes \Omega^1_X(\log D)$. For an irreducible component D_i of D, the residue $\nabla_i \in End_{\mathcal{O}_{D_i}}(\mathcal{E}_x \otimes \mathcal{O}_{D_i})$ is the endomorphism induced by $(id \otimes res_i) \circ \nabla : \mathcal{E}_X \to \mathcal{E}_X \otimes \mathcal{O}_{D_i}$. Let k_i be the constant field of D_i. Then the eigenpolynomial $\Phi_i(T) = \det(T - \nabla_i)$ is a polynomial with coefficient in k_i of degree $r = \mathrm{rank}\, \mathcal{E}_X$. Let $\sum_i = \{\sigma : k_i \to \boldsymbol{C}\}$ be the set of k-morphisms and $s_{\sigma l}(1 \leq l \leq r)$ be the solutions of $\sigma(\Phi_i(T)) = 0$ in $\boldsymbol{C}$ counted with multiplicities. By changing the lattice $\mathcal{E}_X$ if necessary, we may assume $s_{\sigma l} \notin 0,1,2,\ldots$ for all σ and l. The product

$$\Gamma(-\nabla_i) = \prod_{\sigma\in\Sigma_i} \prod_{l=1}^{r} \Gamma(-s_{\sigma l}) \in \boldsymbol{C}^\times / k^\times$$

is determined by the restriction $(\mathcal{E}, \nabla)$ on U and it is independent of the lattice $\mathcal{E}_X$ on X. Let $D_i^* = D_i - \cup_{j\neq i} D_j$ and c_i be the Euler chracteristic of $D_i^* \otimes_{k_i} \boldsymbol{C}$.

Theorem. *Let $\mathcal{M} \in M_k(U, F)$ and $1 = ((\mathcal{O}_U, d), F, 1)$ be the identity object of $M_k(U, F)$. Assume the compactification X of U is projective. Then we have*

$$per(\mathcal{M}) / per(1)^{\mathrm{rank}\mathcal{M}} = (\det \mathcal{M}, c_{X\bmod D}) \times \prod_{i\in I} \Gamma(-\nabla_i)^{-c_i}$$

in $k^\times \backslash \boldsymbol{C}^\times / F^\times$.

Proof will be given somewhere else, the rough idea is as follows. Along the same lines as in [2], by taking a Lefschetz pencil and by induction on dimension of X, it is reduced to the case $X = \boldsymbol{P}^1$, which is proved in [3].

References

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