26. *Regularity Theorems for Holonomic Modules*

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0. Introduction. The regularity theorem for a system of ordinary linear differential equations has a long history. Malgrange [17] has shown the regular singularity of the system is equivalent to the convergence of its formal power series solutions. Ramis [18] extended the results to the irregular singular case, that is, the irregularity is characterized by the Gevrey growth order of its formal power series solutions. In the real case, Komatsu [15] obtained a similar result comparing ultra-distribution and hyperfunction solutions.

One of the important problems is to extend these results to the higher dimensional case. The deep study of holonomic systems due to Kashiwara-Kawai [10] and Kashiwara [9] established the regularity theorems for holonomic modules in the regular singular case. The purpose of this paper is to give several regularity theorems for the irregular holonomic modules.

1. Preliminary. Let X be a complex manifold of dimension n and π : $T^*X \to X$ its cotangent bundle. Set $\mathring{T}^*X = T^*X \setminus T^*_XX$ and denote by $\mathring{\pi}$ the restriction of π to $\mathring{T}^*X$. We choose a local coordinate system of X as $(x_1, \dots, x_n)$ and that of T^*X as $(x_1, \dots, x_n ; \xi_1, \dots, \xi_n)$. T^*X is endowed with the sheaf $\mathscr{E}_X^\infty$ of micro-differential operators of infinite order constructed by Sato-Kawai-Kashiwara [19].

We denote by $\mathscr{E}_X$ (resp. $\mathscr{E}_X(m)$) the subsheaf of $\mathscr{E}_X^\infty$ consisting of micro-differential operators of finite order (resp. micro-differential operators of order at most m). For the theory of $\mathscr{E}_X$, see [19] and Schapira [20]. Now we define the subsheaf $\mathscr{E}_X^{(s)}$ of micro-differential operators of Gevrey growth order (s) for any $s \in (1, \infty)$.

Definition 1.1. *For an open subset U of T^*X, a sum $\sum_{i \in \mathbf{Z}} P_i(x, \xi) \in \mathscr{E}_X^\infty(U)$ belongs to $\mathscr{E}_X^{(s)}(U)$ if and only if $\{P_i\}_{i \in \mathbf{N}}$ satisfies the following estimate* (1.1); *for any compact subset K of U, there exists a positive constant C_K such that*

$$\sup_K | P_i(x, \xi) | \leq \frac{C_K^i}{i!^s} \quad (i \geq 0). \tag{1.1}$$

For convenience, we set $\mathscr{E}_X^{(1)} := \mathscr{E}_X^\infty$ and $\mathscr{E}_X^{(\infty)} := \mathscr{E}_X$.

Next we review briefly the definition of the sheaf of holomorphic microfunctions in Gevrey class. Let Y be a complex submanifold of X and T^*_YX its conormal bundle. Then we define the subsheaf $\mathscr{C}_{Y|X}^{\mathbf{R},(s)}$ of the holomorphic microfunctions $\mathscr{C}_{Y|X}^{\mathbf{R}}$ as

$$\mathscr{C}_{Y|X}^{\mathbf{R},(s)} := \mathscr{E}_X^{(s)} \mathscr{C}_{Y|X}^{\mathbf{R},f}$$

where $\mathscr{C}_{Y|X}^{\boldsymbol{R},f}$ is the sheaf of tempered holomorphic microfunctions (for the definition, see Andronikof [1,2]). Remark that these sheaves are also defined by the functor $T-\mu.^{(s)}(\mathscr{O})$, which is a natural extension of tempered microlocalization functor $T-\mu.(\mathscr{O})$ constructed by Andronikof [1,2],

$$\mathscr{E}_X^{(s)} := \tau^{-1}\tau_* T - \mu_X^{(s)}(\mathscr{O}_{X\times X}) \otimes \Omega_X[\dim X]$$

where $\tau : \mathring{T}^*X \to P^*X$ is a canonical projection, and

$$\mathscr{C}_{Y|X}^{\boldsymbol{R},(s)} := T - \mu_Y^{(s)}(\mathscr{O}_X)[\operatorname{codim} Y].$$

For the definition and properties of the Gevrey microlocalization functor, refer to Honda [5].

Let V be a regular or maximally degenerate involutive submanifold of codimension $d \geq 1$ in $\mathring{T}^*X := T^*X \setminus T_X^*X$. We define the subsheaf $I_V \subset \mathscr{E}_X(1)$ by

$$I_V := \{P \in \mathscr{E}_X(1)\ ;\ \delta_1(P)\,|_V \equiv 0\}.$$

Here we denote the symbol map of degree 1 by $\delta_1(\cdot)$. Now we define the sheaf of rings $\mathscr{E}_V^{(\sigma)}$ in $\mathring{T}^*X$ for a rational number $\sigma \in [1, \infty)$.

$$\mathscr{E}_V^{(\sigma)} := \sum_{n\geq 0} \mathscr{E}_X\left(\frac{(1-\sigma)n}{\sigma}\right) I_V^n.$$

In case $\sigma = 1$, this sheaf coincides with the sheaf $\mathscr{E}_V$ defined in Kashiwara-Oshima [10] and [12].

We list up some main properties of the sheaf $\mathscr{E}_V^{(\sigma)}$.

(1) $\mathscr{E}_V^{(\sigma)}$ is a subring of $\mathscr{E}_X$.

(2) $\mathscr{E}_X(0) \subset \mathscr{E}_V^{(\sigma)}$, and $\mathscr{E}_V^{(\sigma)}$ is a left and right $\mathscr{E}_X(0)$ module.

(3) $\mathscr{E}_V^{(\sigma)}$ is a sheaf of Noetherian ring, and any coherent $\mathscr{E}_X$ module is pseudocoherent over $\mathscr{E}_V^{(\sigma)}$.

(4) If $P \in \mathscr{E}_V^{(\sigma)}$, then its formal ajoint operator P^* belongs to $\mathscr{E}_V^{(\sigma)}$.

Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$ modules in a neighborhood of $p \in \mathring{T}^*X$. We first define the weak irregularity of $\mathscr{M}$ at a smooth point of its support. Given $p \notin \operatorname{supp}(\mathscr{M})_{sing} \cup T_X^*X$.

Definition 1.2. $\mathscr{M}$ has weak irregularity at most σ at p if and only if $\mathscr{M}$ satisfies the following conditions.

There exist an open neighborhood U of p, maximally degenerate involulive submanifold V with its singular locus $\operatorname{supp}(\mathscr{M})$, and an $\mathscr{E}_V^{(\sigma)}$ module $\mathscr{M}_0$ on U which generates $\mathscr{M}$ over $\mathscr{E}_X$ and is finitely generated over $\mathscr{E}_X(0)$ at any point of a dense subset in $\operatorname{supp}(\mathscr{M}) \cap U$.

Next we define weak irregularity in the general case.

Definition 1.3. (1) A holonomic $\mathscr{E}_X$ module $\mathscr{M}$ has weak irregularity at most σ at p if and only if there exist an open neighborhood U of p and a closed analytic subset $Z \supset \operatorname{supp}(\mathscr{M})_{sing}$ with $\operatorname{codim} Z \geq \dim X + 1$ such that $\mathscr{M}$ has weak irregularity at most σ at any point in $U \setminus Z \cap \mathring{T}^*X$.
(2) A holonomic $\mathscr{D}_X$ module $\mathscr{N}$ has weak irregularity at most σ if and only if $\mathscr{E}_X \otimes_{\mathscr{D}_X} \mathscr{N}$ has irregularity at most σ at any point in $\mathring{T}^*X$.

2. Statement of main theorem.

Main theorem. *Let $U \subset T^*X$ be a $\boldsymbol{C}^\times$ conic open set, $\mathscr{M}$ a holonomic $\mathscr{E}_X$ modules on U and $\sigma \geq 1$ a rational number. Then the following conditions* (1), *(2) and (3) are equivalent.*

(1) There exists a holonomic $\mathscr{E}_X$ module $\mathcal{M}_{reg}$ with regular singularities satisfying

$$\mathscr{E}_X^{(s)} \otimes_{\mathscr{E}_X} \mathcal{M} \simeq \mathscr{E}_X^{(s)} \otimes_{\mathscr{E}_X} \mathcal{M}_{reg}$$

in U for all $s \in \left[1, \frac{\sigma}{\sigma-1}\right]$.

(2) For any submanifold $Y \subset X$ and any $s \in \left[1, \frac{\sigma}{\sigma-1}\right]$, we have

$$\boldsymbol{R}\,\mathrm{Hom}_{\mathscr{E}_X}(\mathcal{M}, \mathscr{C}_{Y|X}^{\boldsymbol{R},(s)})\,|_U \simeq \boldsymbol{R}\,\mathrm{Hom}_{\mathscr{E}_X}(\mathcal{M}, \mathscr{C}_{Y|X}^{\boldsymbol{R}})\,|_U.$$

(3) $\mathcal{M}$ has weak irregularity at most σ in U.

Sketch of proof. (3) implies (1), as was shown in [4].

We will show (1) implies (2). Employing a quantized contanct transformation, we may assume $Y=\{x_1=0\}$ and $p=(0\,;dx_1)$. On account of the condition (1), it is enough to show that for a holonomic right $\mathscr{D}_X$ module $\mathcal{M}$ with regular singularity,

$$\mathcal{M} \overset{\boldsymbol{L}}{\otimes}_{\mathscr{D}_X} \frac{\mathscr{C}_{Y|X_p}^{\boldsymbol{R}}}{\mathscr{C}_{Y|X_p}^{\boldsymbol{R},(s)}} = 0.$$

Now we define a map $f\ : X \to X$ by $x_1 = x_1^N$ and $x_j = x_j (j \geq 2)$ for a positive integer N. Then the classical ramification method (cf. [11 ; Lemma 4.1.5]) implies for a large N,

$$\mathrm{char}(H^0\boldsymbol{L}f^*\mathcal{M}) \subset Y \underset{X}{\times} T^*X \tag{2.1}$$

in a neighborhood of p, and

$$\mathrm{supp}(H^k\boldsymbol{L}f^*\mathcal{M}) \subset Y$$

for any $k \geq 1$. Moreover we have the isomorphism

$$\mathcal{M}_{\pi(p)} \overset{\boldsymbol{L}}{\otimes}_{\mathscr{D}_X} \frac{\mathscr{C}_{Y|X_p}^{\boldsymbol{R}}}{\mathscr{C}_{Y|X_p}^{\boldsymbol{R},(s)}} \simeq (\boldsymbol{L}f^*\mathcal{M})_{\pi(p)} \overset{\boldsymbol{L}}{\otimes}_{\mathscr{D}_X} C \tag{2.2}$$

where C is the $\mathscr{E}_{X_p}$ module induced from $\frac{\mathscr{C}_{Y|X_p}^{\boldsymbol{R}}}{\mathscr{C}_{Y|X_p}^{\boldsymbol{R},(s)}}$ by a formal coordinate change with the map f. Under the situation (2.1), we may assume $\boldsymbol{L}f^*\mathcal{M}$ has a simple form on account of [12; Theorem 3.1], and we can show the right handside of (2.2) is equal to zero by a calculation. By reducing the problem to one dimensional case on account of the Cauchy formula for $\mathscr{E}_X$ modules. We can prove that (2) implies (3). For the details of the proof, see [6].

Using the same technique as above, we can prove the following corollary.

Corollary 1. *Let M be a real analytic manifold with its complexification X. If $\mathcal{M}$ be a holonomic $\mathscr{E}_X$ modules at $p \in T^*X$ with weak irregularity at most σ, then we have the isomorphism for all $s \in \left[1, \frac{\sigma}{\sigma-1}\right]$,*

$$\boldsymbol{R}\,\mathrm{Hom}_{\mathscr{E}_X}(\mathcal{M}, \mathscr{C}_M^{(s)}) \simeq \boldsymbol{R}\,\mathrm{Hom}_{\mathscr{E}_X}(\mathcal{M}, \mathscr{C}_M)$$

where $\mathscr{C}_M$ and (resp. $\mathscr{C}_M^{(s)}$) is the sheaf of microfunctions (resp. microfunctions of Gevrey class (s)).

In the case that $\mathcal{M}$ is regular singular and the solution sheaf is tempered microfunctions, this result is already obtained by Andronikof [3]. Finally we remark that applying the functor τ_* to the result (2) of the main theorem, we can recover the results of Laurent [16].

Corollary 2 [16]. *Let $\mathcal{M}$ be a holonomic $\mathscr{E}_X$ modules at $p \in T^*X$ with weak irregularity at most σ. Then we have the following isomorphisms for all $s \in \left[1, \frac{\sigma}{\sigma-1}\right]$ and for any submanifold $Y \subset X$,*

$$\boldsymbol{R}\,\mathrm{Hom}_{\mathscr{E}_X}(\mathcal{M}, \mathscr{C}^{(s)}_{Y|X}) \simeq \boldsymbol{R}\,\mathrm{Hom}_{\mathscr{E}_X}(\mathcal{M}, \mathscr{C}^{\infty}_{Y|X}).$$

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