61. A Note on Jacobi Sums. III

By Takashi ONO

Department of Mathematics, The Johns Hopkins University, U.S.A.

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This is a continuation of [1] which will be referred to as (II). In this paper, we shall reprove Theorem 2 of (II)[1] in a setting which suggests us a direction in further studies inspired by Stickelberger's theorem. We follow, in general, notation and conventions of (II). This paper is logically independent of (II).

§1. Quotient space $H(\mathfrak{P}^{\omega})$. Let K/k be a finite Galois extension of number fields K, k of finite degree over $\boldsymbol{Q}$ with the Galois group $G = G(K/k)$. Let Π be the set of prime ideals $\mathfrak{P}$ of K unramified for K/k. We shall call a map $\varphi : \Pi \to K^{\times}$ a function of type (S) if it satisfies the following conditions:

(S.1) $\varphi(\mathfrak{P}^s) = \varphi(\mathfrak{P})^s$ for all $s \in G$,

(S.2) there is an $\omega_\varphi \in \boldsymbol{Z}[G]$ such that $(\varphi(\mathfrak{P})) = \mathfrak{P}^{\omega_\varphi}$ for all $\mathfrak{P} \in \Pi$.

Using a prime $\mathfrak{p}$ of k which splits completely in K, one sees that ω_φ is well-defined by φ and that ω_φ belongs to the center $\boldsymbol{Z}[G]_0$ of $\boldsymbol{Z}[G]$. If we denote by Φ the set of all maps φ of type (S), then Φ becomes a multiplicative group in an obvious way and the map $\varphi \to \omega_\varphi$ becomes a homomorphism of Φ into the additive group of $\boldsymbol{Z}[G]_0$ whose kernel consists of all maps $\varphi : \Pi \to \mathfrak{o}_K^{\times}$, the group of units of $\mathfrak{o}_K$.

As in (II), for $\varphi \in \Phi$, $\omega \in \boldsymbol{Z}[G]$, we put

$$(1.1)\qquad \begin{aligned} G(\varphi(\mathfrak{P})) &= \{s \in G ; \varphi(\mathfrak{P})^s = \varphi(\mathfrak{P})\}, \\ G^*(\varphi(\mathfrak{P})) &= \{s \in G ; (\varphi(\mathfrak{P}))^s = (\varphi(\mathfrak{P}))\}, \\ G(\mathfrak{P}^{\omega}) &= \{s \in G ; (\mathfrak{P}^{\omega})^s = \mathfrak{P}^{\omega}\}. \end{aligned}$$

Note that we use the convention $\mathfrak{P}^{st} = (\mathfrak{P}^t)^s$, $s, t \in G$. Since $\omega_\varphi \in \boldsymbol{Z}[G]_0$ we have, by (S.2),

$$(1.2)\qquad G(\mathfrak{P}^{\omega_\varphi}) = G^*(\varphi(\mathfrak{P})) \supset G(\varphi(\mathfrak{P})) \supset G(\mathfrak{P})$$

where $G(\mathfrak{P})$ means the decomposition group of $\mathfrak{P}$, i.e., $G(\mathfrak{P}) = G(\mathfrak{P}^1)$, $1 \in \boldsymbol{Z}[G]$. For an $\omega \in \boldsymbol{Z}[G]_0$, we shall put

$$(1.3)\qquad H(\mathfrak{P}^{\omega}) = G(\mathfrak{P}^{\omega}) / G(\mathfrak{P}).$$

Write an $\omega \in \boldsymbol{Z}[G]_0$ as

$$(1.4)\qquad \omega = \sum_{t \in G} a(t) t.$$

Since $a = a(t)$ is a class function on G, its Fourier expansion makes sense:

$$(1.5)\qquad a = \sum_{\chi \in Irr(G)} a_\chi \chi$$

where $Irr(G)$ denotes the set of $\boldsymbol{C}$-irreducible characters of G. The Fourier coefficients are

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1) As for the statement, see the last line of this paper before Acknowledgement.

(1.6) $$a_\chi = \frac{1}{|G|}\sum_{t\in G} a(t)\bar{\chi}(t), \quad \chi \in Irr(G).$$

In order to describe the quotient space (1.3) in terms of characters, write

(1.7) $$\omega = \sum_{t\in G} a(t)t = \sum_{t\in G/G(\mathfrak{P})}\ \sum_{u\in G(\mathfrak{P})} a(tu)tu.$$

Then we have

(1.8) $$\mathfrak{P}^\omega = \prod_{t\in G/G(\mathfrak{P})} (\mathfrak{P}^t)^{R(t)} \quad \text{with } R(t) = \sum_{u\in G(\mathfrak{P})} a(tu).$$

Since $s\omega = \sum_{t\in G} a(s^{-1}t)t$, we have, by (1.8),

(1.9) $$\mathfrak{P}^{s\omega} = \prod_t (\mathfrak{P}^t)^{R(s^{-1}t)}.$$

By the uniqueness of the prime decomposition of ideals, we obtain, from (1.1), (1.8), (1.9),

(1.10) $$s \in G(\mathfrak{P}^\omega) \Leftrightarrow R(s^{-1}t) = R(t) \quad \text{for all } t \in G.$$

Since $R(t) = \sum_{u\in G(\mathfrak{P})} a(tu) = \sum_u \sum_\chi a_\chi \chi(tu)$, we have, by (1.10),

(1.11) $$s \in H(\mathfrak{P}^\omega) \Leftrightarrow \sum_\chi a_\chi \sum_{u\in G(\mathfrak{P})} (\chi(s^{-1}tu) - \chi(tu)) = 0,\ t \in G,$$

where, by abuse of notation, we identified $s \in G(\mathfrak{P}^\omega)$ with s mod $G(\mathfrak{P})$ in $H(\mathfrak{P}^\omega)$. Hoping (1.11) as a starting step for a nonabelian theory, in the sequel, we shall restrict ourselves to the case of abelian extensions K/k.

§2. Abelian extensions. Notation being as in §1, assume that K/k is abelian. Then (1.11) may be written:

(2.1) $$\sum_{\chi\in\hat{G}} a_\chi(\chi(s^{-1}) - 1)\chi(t) \sum_{u\in G(\mathfrak{P})} \chi(u) = 0 \quad \text{for all } t \in G.$$

By the orthogonality of characters on groups $G(\mathfrak{P})$ and $G/G(\mathfrak{P})$, one sees that (2.1) is equivalent to

(2.2) $$a_\chi(\chi(s) - 1) = 0 \quad \text{for all } \chi \in \widehat{G/G(\mathfrak{P})},$$

or to

(2.3) $$\chi(s) = 1 \quad \text{for all } \chi \in \widehat{G/G(\mathfrak{P})} \text{ such that } a_\chi \neq 0.$$

In view of (1.6), we get

(2.4) $$H(\mathfrak{P}^\omega) = \{s \in G/G(\mathfrak{P})\,;\, \chi(s) = 1 \text{ for all } \chi \in \widehat{G/G(\mathfrak{P})} \text{ such that } \sum_{t\in G} a(t)\bar{\chi}(t) \neq 0\}.$$

§3. Back to the l-th cyclotomic field. Let l be an odd prime and let $k = \boldsymbol{Q}(\zeta)$ be the l-th cyclotomic field, $\zeta = e^{2\pi i/l}$. For a prime $p \neq l$, let $\mathfrak{p}$ be a prime ideal in k such that $\mathfrak{p}\,|\,p$.[2)] We may identify $G = G(k/\boldsymbol{Q})$ with the cyclic group $\boldsymbol{F}_l^\times = \langle w\rangle$ as usual. Thus, for an $\omega = \sum_{t\in\boldsymbol{F}_l^\times} a(t)\sigma_t \in \boldsymbol{Z}[G]$, (2.4) can be written as

(3.1) $$H(\mathfrak{p}^\omega) = \{s \in \boldsymbol{F}_l^\times/(\boldsymbol{F}_l^\times)^g\,;\, \chi(s) = 1 \text{ for all } \chi \in \widehat{\boldsymbol{F}_l^\times/(\boldsymbol{F}_l^\times)^g} \text{ such that } \sum_{t\in\boldsymbol{F}_l^\times} a(t)\bar{\chi}(t) \neq 0\}.$$

Now choose for ω an element in $\boldsymbol{Z}[G]$ with

(3.2) $$a(t) = res_l(t^*), \quad t^* = -t^{-1}$$

2) Note that $l - 1 = f\cdot g$, $N\mathfrak{p} = p^f$, $g = |G/G(\mathfrak{p})|$.

and for χ the character of $\boldsymbol{F}_l^\times/(\boldsymbol{F}_l^\times)^g$ determined by $\chi(w)=e^{\frac{2\pi i}{g}}$. Then we have $\chi(-1)=\chi(w^{\frac{l-1}{2}})=\chi(w)^{\frac{l-1}{2}}=(e^{\frac{2\pi i}{g}})^{\frac{fg}{2}}=(-1)^f$; hence, χ is an *odd* character of $\boldsymbol{F}_l^\times$ if and only if f is odd. Furthermore, we have $\sum_{t\in \boldsymbol{F}_l^\times} a(t)\,\bar{\chi}(t)=\sum_t res_l(t^*)\bar{\chi}(t)=\sum_t res_l(t)\bar{\chi}(t^*)=(-1)^f\sum_t res_l(t)\chi(t)=(-1)^f\sum_{\nu=1}^{l-1}\nu\chi(\nu)$, which is $\neq 0$ if f is odd because $0\neq L(1,\bar{\chi})=\frac{\pi i}{l^2}\tau(\bar{\chi})\sum_{\nu=1}^{l-1}\nu\chi(\nu)$ for any odd character of $\boldsymbol{F}_l^\times$.

Let $s=w^\xi$ be any element in $H(\mathfrak{p}^\omega)$. Since the above odd character χ satisfies the condition in (3.1), we must have $1=\chi(s)=\chi(w)^\xi=e^{\frac{2\pi i}{g}\xi}$; hence $g\mid\xi$, so $s \bmod (\boldsymbol{F}_l^\times)^g=1$. In other words, $H(\mathfrak{p}^\omega)=1$. Now let $J(\mathfrak{p})$ be the Jacobi sum considered in (II), i.e., the one such that $J(\mathfrak{p})=g(\mathfrak{p})^l$, $g(\mathfrak{p})$ being the Gauss sum. By the Stickelberger's theorem $J=J(\mathfrak{p})$ is a function of type (S) for the extension $k/\boldsymbol{Q}$ for which $\omega_J=\omega=\sum_t res_l(t^*)\sigma_t$. Since $H(\mathfrak{p}^\omega)=1$, i.e., $G(\mathfrak{p})=G(\mathfrak{p}^\omega)=G^*(J(\mathfrak{p}))$, we have, by (1.2),

$$\boldsymbol{Q}(J(\mathfrak{p}))=\boldsymbol{Q}(\mathfrak{p}) \text{ if } f \text{ is odd.}^{3)} \text{ (Theorem 2 of (II)).}$$

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3) $\boldsymbol{Q}(\mathfrak{p})$ denotes the decomposition field of $\mathfrak{p}$: $\boldsymbol{Q}(\mathfrak{p})=k^{G(\mathfrak{p})}$.