32. *On Birational Models of Enriques Surfaces in* $\mathbf{P}^3$

By Yumiko UMEZU

Department of Mathematics, School of Medicine, Toho University

(Communicated by Heisuke HIRONAKA, M. J. A., May 12, 1994)

Abstract: We prove that an Enriques surface S with certain configuration of halfpencils is birationally equivalent to a normal quintic surface. No Enriques surfaces are known which do not satisfy this condition. We construct birational maps explicitly, and obtain the defining equations with ten parameters of the image in $\boldsymbol{P}^3$.

Key words: Enriques surface; birational map; normal surface.

§0. Introduction. Let S be an Enriques surface defined over an algebraically closed field k of characteristic 0. By definition S is a non-singular projective surface with $q(S) = p_g(S) = 0$ and $2K_S \sim 0$. An effective divisor E on S is called a *halfpencil* if $|2E|$ is base point free and defines an elliptic fibration (cf. [2]). If E is a halfpencil, then there exists a unique halfpencil E' adjoint with E so that $E' \neq E$ and $|2E'| = |2E|$.

Cossec [3] proved that every Enriques surface admits a birational morphism onto a surface of degree 10 in $\boldsymbol{P}^5$ with isolated rational double points, and also that every Enriques surface is birationally equivalent to a (non-normal) sextic surface in $\boldsymbol{P}^3$.

Theorem 1. *Suppose there exist on* S *three halfpencils* E_1, E_2, E_3 *such that*:

(i) $E_1E_2 = E_2E_3 = E_3E_1 = 1$,

(ii) $(E_1 + E_1') \cap E_2 \cap E_3 = \phi$, *where* E_1' *denotes the halfpencil adjoint with* E_1.

Then S *is birationally equivalent to a normal quintic surface in* $\boldsymbol{P}^3$.

Cossec and Dolgachev [4] defined the *non-degeneracy invariant* $d(S)$ of an Enriques surface S, which is reformulated as follows:

$$d(S) = \max\left\{ r \,\middle|\, \begin{array}{l} \text{There exist on } S \text{ halfpencils } E_1, \ldots, E_r \\ \text{such that } E_iE_j = 1 (1 \leq i \neq j \leq r) \end{array} \right\}.$$

Corollary. *If* $d(S) \geq 4$, *then* S *is birationally equivalent to a normal quintic surface.*

Remark. (1) $d(S) = 10$ for generic S ([3], [1], [5]).

(2) $d(S) \geq 3$ for any S ([3]).

(3) The value $d(S)$ is calculated for S with finite automorphism group classified in [7]. They all have $d(S) \geq 4$ ([8]).

This note is an announcement of the main results in [11]. After the author had written up [11], she received a paper of Y. Kim [6], in which he claims that the condition (ii) of Theorem 1 is satisfied under other assumptions, and hence every Enriques surface is birationally equivalent to a normal quintic surface; his proof is not complete, however. He says he is preparing a corrected version.

§1. Construction of birational maps in Theorem 1. Lemma 1. *Let E_1 and E_2 be effective divisors on a smooth surface such that:*

(i) *E_1 and E_2 have no (-1)-curve as a component,*

(ii) *$|m_1E_1|$ and $|m_2E_2|$ define respective elliptic fibrations for some m_1, $m_2 \geq 1$,*

(iii) $E_1E_2 = 1$.

Then E_1 and E_2 have no common components.

Set $\bar{D} = E_1 + E_1' + E_2 + E_3$. Then, by Lemma 1, $\bar{D}$ has five double points $P_1,\dots,P_5$ other than singular points of E_1, E_1', E_2 and E_3. Let $\tilde{S} \to S$ be the blowing-up at $P_1,\dots,P_5$, and $\tilde{H}$ the proper transform to $\tilde{S}$ of a general member of the linear system $|\bar{D} + K_S - P_1 - \cdots - P_5|$. Then we can show that $|\tilde{H}|$ defines a birational morphism from $\tilde{S}$ onto a normal quintic surface.

The singularity of the image consists of i) four minimally elliptic singularities ([9]), which correspond to E_1, E_1', E_2, E_3, and possibly ii) some rational double points, which correspond to connected components of the sum of all curves on S disjoint from $\bar{D}$.

§2. Proof of Corollary. Lemma 2. *Let E_1, E_2, E_3 be halfpencils on an Enriques surface S such that $E_1E_2 = E_2E_3 = E_3E_1 = 1$. Let E_i' denote the halfpencil adjoint with $E_i (i = 1, 2, 3)$. If E_1, E_2, E_3 meet at one point, then so do E_1, E_2', E_3'; E_1', E_2, E_3'; E_1', E_2', E_3 respectively.*

Let E_1, E_2, E_3, E_4 be any halfpencils on S with $E_iE_j = 1 (1 \leq i \neq j \leq 4)$. If E_1, E_2, E_i do not satisfy the hypothesis (ii) of Theorem 1 for $i = 3$, 4, then we can deduce from Lemma 2 that $E_3E_4 \geq 2$, which is a contradiction.

§3. Defining equations. By finding out a relation of degree 5 of a basis of $H^0(\tilde{S}, O_{\tilde{S}}(\tilde{H}))$, we obtain:

Theorem 2. *Let X be a normal quintic surface in $\boldsymbol{P}^3$, which is constructed in the way of §1 from an Enriques surface S satisfying the hypothesis of Theorem 1. Then, with suitable homogeneous coordinates $(X_0 : X_1 : X_2 : X_3)$ of $\boldsymbol{P}^3$, the defining equation of X is of the following form:*

$$
\begin{aligned}
(*) \quad F = {} & X_0^3X_1^2 + X_0^2X_1^3 + X_0^2X_1X_2^2 + X_0^2X_1X_3^2 + c_1X_0^2X_1^2X_2 + c_2X_0^2X_1^2X_3 \\
& + c_3X_0^2X_1X_2X_3 + c_4X_0X_1^2X_2^2 + c_5X_0X_1^2X_3^2 + c_6X_0X_2^2X_3^2 \\
& + c_7X_0X_1^2X_2X_3 + c_8X_0X_1X_2^2X_3 + c_9X_0X_1X_2X_3^2 + c_{10}X_1X_2^2X_3^2 \\
= {} & 0
\end{aligned}
$$

$$(c_4, c_5, c_6, c_{10} \neq 0, c_1, \dots, c_{10} \in k).$$

Remark. The equation $(*)$ is found in [10].

Theorem 3. *Let $F = 0$ be an equation of the form $(*)$. Set $X = \{F = 0\} \subset \boldsymbol{P}^3$. Assume that every point of X, except for $(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$ and $(0:0:0:1)$, is at worst an isolated rational double point. Then X is birationally equivalent to an Enriques surface S which satisfies the* *hypothesis of Theorem* 1, *and* X *is constructed from* S *in the way described in* §1.

References

[1] Barth, W., and Peters, C.: Automorphisms of Enriques surfaces. Invent. Math., **73**, 383–411 (1983).
[2] Barth, W., Peters, C., and Van de Ven, A.: Compact complex surfaces (Ergeb. Math. Grenzgeb., 3. Folge, Bd. 4). Berlin, Heidelberg, New York, Tokyo, Springer (1984).
[3] Cossec, F.: On the Picard group of Enriques surfaces. Math. Ann., **271**, 577–600 (1985).
[4] Cossec, F., and Dolgachev, I. V.: Enriques Surfaces. I (Progr. Math. 76). Birkhäuser, Boston, Basel, Berlin (1989).
[5] ——: Smooth rational curves on Enriques Surfaces. Math. Ann., **272**, 369–384 (1985).
[6] Kim, Y.: On normal quintic Enriques surfaces. Thesis, Univ. of Michigan (1991).
[7] Kondō, S.: Enriques surfaces with finite automorphism groups. Japan J. Math., **12**, 191–282 (1986).
[8] ——: Personal communication.
[9] Laufer, H.: On minimally elliptic singularities. Amer. J. Math., **99**, 1257–1295 (1977).
[10] Stagnaro, E.: Constructing Enriques surfaces from quintics in $\boldsymbol{P}^3_K$. Algebraic Geometry-open Problems (eds. C. Ciliberto *et al.*). Ravello 1982 (Lect. Notes Math., vol. 997, pp. 400–403), Berlin, Heidelberg, New York, Tokyo, Springer (1983).
[11] Umezu, Y.: Normal quintic surfaces which are birationally Enriques surfaces (preprint).