28. *Holomorphic Structure of the Arithmetic-geometric Mean of Gauss. II*

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1. Introduction and the main result. Let us review some definitions and properties of the complex arithmetic-geometric mean. Let a and b be two complex numbers satisfying

(A) $$ab \neq 0, \quad a \pm b \neq 0.$$

A sequence $\{(a_n, b_n)\}$ $(n = 0,1,\dots)$ is called an *agm-sequence* for (a, b) if it satisfies the algorithm,

$$a_0 = a, \quad b_0 = b$$

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, \quad b_n = (a_{n-1}b_{n-1})^{1/2}, \quad n = 1,2,\dots$$

Since there are two possible values of b_n at each step of the algorithm, there are infititely many such agm-sequences for fixed (a, b). We call (a_n, b_n) $(n \geq 1)$ of an agm-sequence *the right choice*, if

$$\mathrm{Re}\left(\frac{b_n}{a_n}\right) > 0 \text{ or } \mathrm{Re}\left(\frac{b_n}{a_n}\right) = 0, \quad \mathrm{Im}\left(\frac{b_n}{a_n}\right) > 0.$$

Note that a_n and b_n also satisfy (A) and that one of (a_n, b_n) and $(a_n, -b_n)$ is always the right choice, while the other is not.

For any agm-sequence $\{(a_n, b_n)\}$ one can prove that both sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit. For "most" of the agm-sequences, however, their limits turn out to be 0. More precisely, for any agm-sequence, its limit $\tau = \lim a_n = \lim b_n$ exists. And $\tau \neq 0$ if and only if (a_n, b_n) is the right choice for all but finitely many $n \geq 1$. An agm-sequence satisfying this condition will be called a *good one*. Let $\mathfrak{M}(a, b)$ denote the set of all the non-zero limits of good agm-sequences for (a, b). *The simplest mean* of (a, b), denoted by $M(a, b)$, is defined as the limit of $\{a_n\}$ of such agm-sequence $\{(a_n, b_n)\}$ for (a, b) that (a_n, b_n) is the right choice for all $n \geq 1$. Naturally we have $M(a, b) \in \mathfrak{M}(a, b)$.

Now a question is raised as to the correspondence between an agm-sequence and its limit. We ask whether the correspondence is one-to-one or not. Our question will be of any interest only for the class of good agm-sequences. If "bad" agm-sequences were included, there would be no one-to-one correspondence, since such a sequence has always 0 as its limit, no matter how it makes various choices. The following theorem answers the above question.

Theorem 1. *Let a and b satisfy the condition* (A). *Suppose that two agm-sequences $\{(a_n, b_n)\}$ and $\{(a'_n, b'_n)\}$ for (a, b) are good ones so that they* *have their limits* $\tau = \lim a_n$ *and* $\tau' = \lim a'_n$ *in* $\mathfrak{M}(a, b)$. *If* $\tau = \tau'$, *then the two agm-sequences are equal, namely*

$$(a_n, b_n) = (a'_n, b'_n) \text{ for all } n \geq 0.$$

In this paper an outline of the proof of Theorem 1 will be given. In the process we will need the result, as well as the proof, of the following theorem.

Theorem 2. *Let a and b satisfy the condition* (A) *and* $|a| \geq |b|$. *Then a complex number τ belongs to $\mathfrak{M}(a, b)$ if and only if*

$$\frac{1}{\tau} = \frac{p}{M(a, b)} + i\frac{q}{M(a + b, a - b)}, \tag{1}$$

where p and q are arbitrary relatively prime integers satisfying

$$p \equiv 1 \bmod 4 \text{ and } q \equiv 0 \bmod 4.$$

Theorem 2, or at least part of it, was already conceived by Gauss, although he never gave an exact statement as above. Different proofs of the theorem were given by several authors (e.g. Geppert [2], Cox[1] and Nishiwada [3, 4]). Since the proofs in [1] and [2] rely heavily on some theta identities and seem unsuitable for our purpose, we prefer to follow the arguments given in [3, 4].

I would like to thank Prof. Masaaki Yoshida for informing me of some facts related to the injectivity of the map (3).

2. Analytic continuation of $M(1, z)$. Due to the homogeneity of $M(a, b)$ and $\mathfrak{M}(a, b)$, we may put $a = 1$, $b = z$. Assumption (A) reads now as

$$z \in \mathbf{C}_0 := \mathbf{C} \setminus \{0, \pm 1\}.$$

An agm-sequence for $(1, z)$ can be written as $\{(a_n(z), b_n(z))\}$, where $a_n(z)$, $b_n(z)$ are (multi-valued) holomorphic functions in $\mathbf{C}_0$. Furthermore, their limit $\tau(z) = \lim a_n(z) = \lim b_n(z)$ defines a germ of a holomorphic function. The following proposition states that all the values $\tau(z) \in \mathfrak{M}(1, z)$ are actually branches of a single holomorphic function, for instance of $M(1, z)$.

Proposition 3 ([3, 4]). *Let $\{(a_n(z_0), b_n(z_0))\}$ be an agm-sequence for $(1, z_0)$, $z_0 \in \mathbf{C}_0$. Suppose that there is a number $N\,(\geq 2)$ such that $(a_n(z_0), b_n(z_0))$ is the right choice for every $n \geq N$. Then there exists a point $z_1 \in \mathbf{C}_0$ and a curve γ in $\mathbf{C}_0$ joining z_0 with z_1 such that $((\gamma)_* a_n(z_1), (\gamma)_* b_n(z_1))$ is the right choice for every $n \geq N - 1$. Here $(\gamma)_*$ denotes the analytic continuation along the curve γ.*

Let us now study those values which are attained by analytic continuation of $M(1, z)$ along various cycles of the fundamental group $\pi_1(\mathbf{C}_0, z)$. Note that this is a free group generated by three cycles γ_{-1}, γ_0 and γ_1, which are defined by some positively oriented Jordan curves through z respectively surrounding -1, 0 and 1.

Both $M(1, z)^{-1}$ and $M(1 + z, 1 - z)^{-1}$ have integral expressions on the elliptic curve, $y^2 = x(1 - x)(1 + (z^2 - 1)x)$, as follows,

$$\frac{1}{M(1, z)} = \frac{1}{\pi}\int_0^1 \frac{dx}{y}, \quad \frac{i}{M(1 + z, 1 - z)} = \frac{1}{\pi}\int_0^{-\infty} \frac{dx}{y}.$$

Using these expressions one can compute circuit matrices of $(M(1, z)^{-1}, iM(1 + z, 1 - z)^{-1})$ along the cycles γ_{-1}, γ_0 and γ_1. We get

(2) $$\mu(\gamma_1) = V,\ \mu(\gamma_0) = U^{-2},\ \mu(\gamma_{-1}) = UVU^{-1},$$
where
$$U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The relations (2) induce the homomorphism

(3) $$\mu : \pi_1(\mathbf{C}_0, z) \to \Gamma \subset SL(2, \mathbf{Z}),$$
where Γ is the subgroup of $SL(2, \mathbf{Z})$ generated by V, U^{-2} and UVU^{-1}. Then the circuit matrix of $(M(1, z)^{-1}, iM(1+z, 1-z)^{-1})$ along an arbitrary cycle $\gamma \in \pi_1(\mathbf{C}_0, z)$ is given by the image $\mu(\gamma)$ of the above map (3).

A standard discussion on modular groups can prove that
$$\Gamma = \Gamma_2(4) := \left\{\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbf{Z})\, ;\, p \equiv s \equiv 1 \bmod 4,\ q \equiv 0 \bmod 4,\ r \equiv 0 \bmod 2\right\}.$$
This shows in particular that for any closed curve γ in $\mathbf{C}_0$ starting at z,

(4) $$(\gamma)_* M(1, z)^{-1} = pM(1, z)^{-1} + iqM(1+z, 1-z)^{-1},$$
where (p, q) is the first row of the matrix $\mu(\gamma)$. Theorem 2 is an immediate consequence of the above equality (4).

3. Outline of the proof of Theorem 1. We may assume that $a = 1$, $b = z_0$ in Theorem 1. Consider two agm-sequences $\{(a_n(z_0), b_n(z_0))\}$ and $\{(a'_n(z_0), b'_n(z_0))\}$ and their respective limits $\tau_1(z_0)$ and $\tau_2(z_0)$, both belonging to $\mathfrak{M}(1, z_0)$. Our proof can be divided into several steps.

Step 1. Our assumption that $\tau_1(z_0) = \tau_2(z_0)$ implies that they actually coincide as germs of holomorphic functions, namely $\tau_1(z) = \tau_2(z)$ near z_0. This can be proved if one notices that $M(1, z)^{-1}$ and $M(1+z, 1-z)^{-1}$ are linearly independent as different periods of an elliptic curve and therefore that the numbers p and q corresponding to $\tau_1(z_0)$ in (1) are equal to those corresponding to $\tau_2(z_0)$.

Step 2. Prop. 3 allows us to find a closed curve ρ_1 in $\mathbf{C}_0$ through z_0, such that the analytic continuation along ρ_1 brings both $\tau_1(z)$ and its defining agm-sequcnce $\{(a_n(z), b_n(z))\}$ to $M(1, z)$ and its agm-sequence consisting only of the right choices. Similarly we can find a closed curve ρ_2 for $\tau_2(z)$ and its agm-sequence.

Step 3. Now the analytic continuation $(\rho_2\rho_1^{-1})_*$ keeps $M(1, z)$ invariant, while its defining agm-sequence might be changed. Its corresponding circuit matrix $\mu(\rho_2\rho_1^{-1})$ by the map (3) must be of the form
$$\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} = V^m,\ m \in \mathbf{Z}.$$
Since the map is injective and $\mu(\gamma_1) = V$, it follows that $\rho_2\rho_1^{-1} = \gamma_1^m$ as elements of $\pi_1(\mathbf{C}_0, z_0)$.

Step 4. One can prove that $M(1, z)$ and all the rightly chosen component functions of its agm-sequence have the point 1 as their removable singularity. Therefore the action of $(\gamma_1)_*^m$ does not change these functions. It follows then that the actions $(\rho_1^{-1})_*$ and $(\rho_2^{-1})_*$ on the agm-sequence defining $M(1, z)$ bring about the same agm-sequence; namely we have
$$\{(a_n(z_0), b_n(z_0))\} = \{(a'_n(z_0), b'_n(z_0))\}.$$

References

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[4] ——: A structural theorem of the complex arithmetic-geometric mean (to appear in Human and Environmental Studies, Kyoto Univ., **3**) (in Japanese).