58. *Selfsimilar Shrinking Curves for Anisotropic Curvature Flow Equations*

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We consider a simple looking ordinary differential equation of the form

$$u'' + u - \frac{a(\theta)}{u} = 0 \text{ in } \boldsymbol{R} \tag{1}$$

with a given positive function $a(\theta)$. This equation arises in describing selfsimilar solutions of anisotropic curvature flow equations. Since θ is the argument of the normal $\vec{n}$ of the curve, it is natural to impose 2π-periodicity for $a(\theta)$ in (1) and to ask for existence and uniqueness of 2π-periodic solutions.

The physical background of the above problem is an evolution equation for embedded closed curves $\{\Gamma_t\}_{t>0}$ in $\boldsymbol{R}^2$ (see [10]):

Consider an equation for Γ_t, where the normal velocity V is given by the curvature k weighted by a direction-dependent factor $a(\theta)$, i.e.

$$V = a(\theta)k, \quad a(\theta) = \beta(\theta)^{-1}(\gamma''(\theta) + \gamma(\theta)),$$

where β and $\gamma'' + \gamma$ are assumed to be positive, so that the equation is parabolic. γ is called the surface energy density and β is called the kinetic coefficient.

In case $a(\theta) \equiv$ const. it is well known (see [3], [4], [6] and [9]) that any initial curve becomes convex, after this it extincts in finite time, and that the type of shrinking is asymptotically similar to that of a shrinking circle $C_t = (2(t_* - t))^{1/2} C$, where C denotes the unit circle centered at the origin. (Here the time t_* is the extinction time and λC denotes the dilation of C with multiplier λ.) The curvature of the circle then is a solution of (1).

In case of more general $a(\theta)$, it was shown in [12] that selfsimilar solutions, i.e. solutions satisfying

$$\Gamma_t = (2(t_* - t))^{1/2}\Gamma$$

and thereby equation (1), exist if $\beta(\theta)\gamma(\theta) =$ const. Then Γ defined as the boundary of the so-called Wulff-Shape W_γ, i.e.

$$W_\gamma := \{x \in \boldsymbol{R}^2 \mid x \cdot \vec{m}(\sigma) \leq \gamma(\sigma) \text{ for all } \sigma \in \boldsymbol{R}\}, \tag{2}$$

yields a solution Γ_t of the evolution problem. Here $\vec{m}(\sigma)$ denotes a unit vector whose argument equals σ.

Our existence result now shows that such selfsimilar solutions exist for arbitrary positive $a(\theta)$. To simplify the notation we notice that a 2π-periodic function can be regarded as a function on the flat torus $\boldsymbol{T} := \boldsymbol{R}/2\pi\boldsymbol{Z}$. Thus we define

$$C_+^2(\boldsymbol{T}) = \{u \in C^2(\boldsymbol{R}) \mid u(\theta + 2\pi) = u(\theta) \text{ for all } \theta \in \boldsymbol{R},\ u > 0\}.$$

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Main existence theorem. *Assume that $a(\theta)$ is a positive, continuous function on $\boldsymbol{T}$. Then there is a function $u \in C_+^2(\boldsymbol{T})$ solving* (1).

The proof is based on a-priori estimates and a continuity method. We can derive a-priori bounds for solutions of (1), that only depend on the bounds of $a(\theta)$ from below and above. This enables us to apply a continuity method connecting the well known case $a(\theta) \equiv$ const. and the case of general $a(\theta)$. For details we would like to refer to [2].

Concerning uniqueness, we unfortunately have to make an additional assumption on $a(\theta)$:

Uniqueness theorem. *Let $a(\theta)$ be a positive, continuous and π-periodic function in $\boldsymbol{R}$. Then the solution of* (1) *is unique.*

The main tools in proving the result is a generalization of an isoperimetric inequality by Gage. This result requires the π-periodicity of $a(\theta)$.

Let us first introduce some notation: We denote the area of a set A by $m(A)$, the interior of a closed curve Γ by $\operatorname{int}\Gamma$, the length of a curve Γ by L and its surface energy with respect to some surface energy density f by

$$F_f(\Gamma) = \int_0^L f(\theta(s))\,ds.$$

Here s denotes the arclength parameter and $\theta(s)$ is the argument of $\vec{n}$ at the point $x(s)$ of the curve. We note also that the area $m(A)$, using integration by parts, can be expressed as an integral over the scalar product of the position vector x and the normal $\vec{n}$, the so-called support function $p(s) = -\langle x(s), \vec{n}(s)\rangle$, i.e.

$$m(A) = \frac{1}{2}\int_0^L p(s)\,ds.$$

Proposition (see [5]). *Let Γ be an arbitrary closed, convex, embedded C^2-curve with curvature k and let the surface energy density f be in C^2 and π-periodic. Then*

$$\int_0^L \frac{a(\theta(s))^2 k(s)^2}{f(\theta(s))}\,ds \geq \frac{m(W_f)}{m(\operatorname{int}\Gamma)} F_f(\Gamma). \tag{3}$$

Moreover equality holds if and only if $\Gamma = \partial W_f$. Here $a = (f'' + f)f$

As we would like to make the proof self-contained, we give the simple derivation of an important identity used below to calculate the isoperimetric quantities of selfsimilar curves, and we also give a lemma on the one to one correspondance of Wulff-shapes and their generating functions.

Lemma 1. *Let Γ be an arbitrary closed, convex, embedded C^2-curve with curvature k and let the surface energy density f be in C^2 with $a = (f'' + f)f$ and allowing a Wulff-shape. Then*

$$F_f(\Gamma) = \int_0^L \frac{p(s)}{f(\theta(s))}\,a(\theta(s))k(s)\,ds. \tag{4}$$

Proof. Inserting $\langle x', x'\rangle = 1$ in the definition of $F_f(\Gamma)$ and integrating by parts we have

$$F_f(\Gamma) = -\int_0^L \langle (f(\theta(s))x'(s))', x(s)\rangle\,ds$$

$$= -\int_0^L f(\theta(s))k(s)\,\langle\vec{n}(s), x(s)\rangle ds + \int_0^L f'(\theta(s))(\langle\vec{n}(s), x(s)\rangle)'ds,$$

due to $x'' = k\vec{n}$ and $\langle\vec{n}, x'\rangle = 0$. Another integration by parts yields

$$F_f(\Gamma) = -\int_0^L (f(\theta(s)) + f''(\theta(s)))k(s)\,\langle\vec{n}(s), x(s)\rangle ds$$
$$= \int_0^L \frac{a(\theta(s))}{f(\theta(s))} k(s)p(s)ds.$$

Lemma 2. *Let $f_i \in C_+^2(\boldsymbol{T})$, $f_i'' + f_i > 0$, $i = 1,2$, and let the Wulff-shapes generated by f_1 and f_2 be identical, i.e. $W_{f_1} = W_{f_2}$. Then $f_1 = f_2$.*

Proof. This follows from elementary facts from convex analysis (see for instance [11]). Define

$$\bar{f}_i(q) = |\,q\,|\, f_i(\theta(q)) \quad \text{for } q \in \boldsymbol{R}^2.$$

Here $\theta(q)$ denotes the argument of q. If $f_i'' + f_i > 0$, then $\bar{f}_i$ is a convex function (see e.g. [8], Appendix B). Moreover the complex conjugate of $\bar{f}_i$

$$\bar{f}_i^*(q^*) := \sup_{q \in \boldsymbol{R}^2} \{\langle q, q^*\rangle - \bar{f}_i(q)\}$$

equals an indicator function of W_f, i.e.

$$\bar{f}_i^*(q^*) = \begin{cases} 0, & \text{if } q^* \in W_{f_i}, \\ \infty, & \text{otherwise} \end{cases}$$

Thus $\bar{f}_1^* = \bar{f}_2^*$ by the assumption, and so also $\bar{f}_1^{**} = \bar{f}_2^{**}$ holds. But as the $\bar{f}_i$ are convex, the second conjugate equals the function itself, which means $f_1 = f_2$.

Proof of the uniqueness result. Suppose there are two solutions, so (1), and consequently two decompositions of $a(\theta)$

$$a(\theta) = (f_i''(\theta) + f_i(\theta))f_i(\theta), \quad i = 1,2.$$

Now let Γ be any selfsimilar solution of $V = a(\theta)k$. Then Γ solves

$$p(s) = -\langle x(s), \vec{n}(s)\rangle = a(\theta(s))k(s). \tag{5}$$

By Lemma 1 and the Gage inequality

$$F_{f_i}(\Gamma) = \int_0^L \frac{a(\theta(s))^2 k(s)^2}{f_i(\theta(s))} ds \geq \frac{m(W_{f_i})}{m(\mathrm{int}\,\Gamma)} F_{f_i}(\Gamma).$$

But the area of Γ is given by

$$m(\mathrm{int}\,\Gamma) = \frac{1}{2}\int_0^L a(\theta(s))k(s)ds = \frac{1}{2}\int_0^{2\pi} a(\theta)d\theta = m(W_{f_i}).$$

Therefore equality holds in the Gage inequality, which is only possible for $\Gamma = W_{f_i}$. Using Lemma 2 we immediately conclude $f_1 = f_2$.

Remarks. (i) The problem (1) was also studied in [5] and [7]. However, they have to assume that a is smooth in order to study a related parabolic partial differential equation. Our proof is more direct and requires only boundedness of $a(\theta)$.

(ii) Another proof of the uniqueness can be given: Suppose there exist two different solutions f and u to (1), the corresponding curves denoted by Γ_f and Γ_u, respectively. Regard f as the new surface energy density. Similar to the above argument one can show that both curves minimize the isoperimetric quantity $F_f(\Gamma)^2 - 4m(W_f)m(\mathrm{int}\,\Gamma)$. So by the Wulff-theorem (in case of curves see for instance [1]) they both must be W_f. Thus $\Gamma_u = \Gamma_f$

and $u = f$.

Although quite similar to the proof given before, this proof makes use of a highly nontrivial result, the Wulff-theorem, whereas the other one uses simple convex analysis instead.

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