Cubic Hyper-equisingular Families of Complex Projective Varieties. II

By Shoji TSUBOI

Department of Mathematics, Kagoshima University

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This is a continuation of our previous paper [4], which will be referred to as Part I in this note. We inherit the notation and terminology of it.

§3. Variations of mixed Hodge structure.

3.1 Theorem. *Let* $\mathscr{X}_{\cdot} \xrightarrow{a_{\cdot}} \mathscr{X} \xrightarrow{\pi} M$ *be an* n-*cubic* $(n \geq 1)$ *hyper-equisingular family of complex projective varieties, parametrized by a complex manifold* M. *We define* $R^{\ell}_{\boldsymbol{Z}}(\pi) := R^{\ell}\pi_{*}\boldsymbol{Z}_{\mathscr{X}}$ *(modulo torsion)* $(0 \leq \ell \leq 2(\dim\mathscr{X} - \dim M))$, $R^{\ell}_{\boldsymbol{Q}}(\pi) := R^{\ell}_{\boldsymbol{Z}}(\pi) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ *and* $R^{\ell}_{\mathscr{O}}(\pi) := R^{\ell}\pi_{*}(\pi^{\cdot}\mathscr{O}_M) \simeq R^{\ell}\pi_{*}(DR^{\cdot}_{\mathscr{X}/M})$, *where* $\pi^{\cdot}\mathscr{O}_M$ *is the topological inverse of the structure sheaf of* M *by the map* $\pi : \mathscr{X} \to M$ *and* $DR^{\cdot}_{\mathscr{X}/M}$ *the cohomological relative de Rham complex of the family* $\pi : \mathscr{X} \to M$. *Then there exist a family of increasing sub-local systems* $\boldsymbol{W}$ *(weight filteration) on* $R^{\ell}_{\boldsymbol{Q}}(\pi)$ *and a family of decreasing holomorphic subbundles* $\boldsymbol{F}$ *(Hodge filteration) on* $R^{\ell}_{\mathscr{O}}(\pi)$ *such that*

(i) there are spectral sequences

$$ {}_{W}E_1^{p,q} \simeq \oplus_{|\alpha|=p+1} R^q\pi_{\alpha *}\boldsymbol{Q}_{\mathscr{X}_\alpha} \Rightarrow $$
$$ {}_{W}E_\infty^{p,q} = Gr^W_{-p}(R^{p+q}_{\boldsymbol{Q}}(\pi)), $$
$$ {}_{F}E_1^{p,q} \simeq \boldsymbol{R}^q\pi_{*}(s(a_{1\cdot *}\Omega^p_{\mathscr{X}_{\cdot}/M})[1]) \Rightarrow $$
$$ {}_{F}E_\infty^{p,q} = Gr^p_F(R^{p+q}_{\mathscr{O}}(\pi)) $$

with ${}_{W}E_2^{p,q} = {}_{W}E_\infty^{p,q}$, ${}_{F}E_1^{p,q} = {}_{F}E_\infty^{p,q}$,

(ii) $(R^{\ell}_{\boldsymbol{Z}}(\pi), \boldsymbol{W}[\ell], \boldsymbol{F})$ *defines mixed Hodge strucutre at each point* $t \in M$, *where* $\boldsymbol{W}[\ell]$ *denotes the shift of the filteration degree to the right* by ℓ, i.e., $\boldsymbol{W}[\ell]_q := \boldsymbol{W}_{q-\ell}$, and

(iii) (the Griffiths transversality)

$$ \nabla \mathscr{F}^p \subset \Omega^1_M \otimes \mathscr{F}^{p-1}, $$

where ∇ denotes the Gauss-Mannin connection on $R^{\ell}_{\mathscr{O}}(\pi)$.

Outline of the proof. (i), (ii): By Theorem 2.1 and Theorem 2.2 in [4], we have an isomorphism

$$ \pi^{\cdot}\mathscr{O}_M \approx DR^{\cdot}{}_{\mathscr{X}/M} \approx s(a_{1\cdot *}\Omega^{\cdot}_{\mathscr{X}_{\cdot}/M})[1] $$

in $D^+(\mathscr{X}, \boldsymbol{C})$, where $a_{1\cdot *}\Omega^{\cdot}_{\mathscr{X}_{\cdot}/M}$ is the n-cubic object of complexes of $\boldsymbol{C}$-vector spaces coming from $\Omega^{\cdot}_{\mathscr{X}_{\cdot}/M}$, and $s(a_{1\cdot *}\Omega^{\cdot}_{\mathscr{X}_{\cdot}/M})$ is its associated single complex (cf. Part I, [1, Exposé I,6]). By this isomorphism we have

$$ R^{\ell}_{\mathscr{O}}(\pi) := R^{\ell}\pi_{*}(\pi^{\cdot}\mathscr{O}_M) \simeq \boldsymbol{R}^{\ell}\pi_{*}(s(a_{1\cdot *}\Omega^{\cdot}_{\mathscr{X}_{\cdot}/M})[1]). $$

To compute the hyper-direct image $\boldsymbol{R}^{\ell}\pi_{*}(s(a_{1\cdot *}\Omega^{\cdot}_{\mathscr{X}_{\cdot}/M})[1])$, we shall use the fine resolution $\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_{\cdot}/M}$ of $\Omega^{\cdot}_{\mathscr{X}_{\cdot}/M}$, where $\mathscr{A}^{r,s}_{\mathscr{X}_\alpha/M}$ are the sheaves of C^{∞} relative differential forms of type (r, s) on $\mathscr{X}_\alpha (\alpha \in \square_n)$. Then the natural homomorphism

$$ s(a_{1\cdot *}\Omega^{\cdot}_{\mathscr{X}_{\cdot}/M})[1] \to s(a_{1\cdot *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_{\cdot}/M})[1] $$

is an isomorphism in $D^+(\mathscr{X}, \boldsymbol{C})$, where $\mathrm{tot}\,\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_\alpha/M}$ is the single complex associated to the double complex $\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_\alpha/M}$ for each $\alpha \in \square_n$. Since $s(a_{1\cdot *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_{\cdot}/M})[1]$ is π_*-acyclic, we have

$$ R^{\ell}_{\mathscr{O}}(\pi) \simeq H^{\ell}(\pi_{*}s(a_{1\cdot *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_{\cdot}/M})[1]). $$

We define an increasing filteration $\boldsymbol{W} = \{W_q\}$ and a decreasing one $F = \{F^q\}$ on the single complex $L := \pi_{*}s(a_{1\cdot *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_{\cdot}/M})$ [1] by

$$ W_{-q}(\pi_{*}s(a_{1\cdot *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_{\cdot}/M})[1]) $$
$$:= \sigma_{|\alpha| \geq q+1}\pi_{*}s(a_{1\alpha *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_\alpha/M}) \quad (q \geq 0) \text{ and} $$
$$ F^p(\pi_{*}s(a_{1\cdot *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_{\cdot}/M})[1]) $$
$$:= \sigma_{k \geq p}\pi_{*}s(a_{1\cdot *}\mathrm{tot}\mathscr{A}^{k\cdot}_{\mathscr{X}_{\cdot}/M})[1] \quad (p \geq 0), $$

where $\sigma_{|\alpha| \geq q+1}\pi_{*}s(a_{1\alpha *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_\alpha/M}) := \sigma_{\geq q}(L)$ if we put $L := \pi_{*}s(a_{1\cdot *}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_{\cdot}/M})[1]$. ($\sigma_{\geq q}$: *stupid filteration*). Notice that the filteration W is defined over $\boldsymbol{Q}$. We calculate the spectral sequence associated to these filterations, abutting to $R^{\ell}_{\mathscr{O}}(\pi)$. Since (L_t, W, F) is a *cohomological mixed Hodge complex* in the sense of Deligne for any $t \in M$ (for definition see [1, (8.1.6)]), the spectral sequence $\{E_r(L_t, W), d_r\}$ degenerates at the E_2-terms and the one associated to $\boldsymbol{F}$ degenerates at the E_1-terms ([2, p.48, Théorème 3.2.1 (Deligne), (vi), (v)]). The assertions (i) and (ii) follow from this.

(iii): We take a point $o \in M$ and put $X_\alpha := (\pi \cdot a_\alpha)^{-1}(o)$, $X := \pi^{-1}(o)$. By the definition of an n-cubic hyper-equisingular family $\mathscr{X}_{\cdot} \xrightarrow{a_{\cdot}} \mathscr{X} \xrightarrow{\pi} M$, it is analytically locally trivial. Hence, schrinking M sufficiently small around o, we are allowed to assume that there is a system of Stein coverings $\mathscr{U}_\alpha := \{U_i^{(\alpha)}\}_{i \in \Lambda_\alpha}$ of X_α $(\alpha \in \square_n^+)$, which is subject to the following requirements:

(1) for each pair (α, β) of elements of $\mathrm{Ob}(\square_n^+)$ with $\alpha \to \beta$ in $\square_n^+$, there is a map $\lambda_{\alpha\beta} : \Lambda_\beta \to \Lambda_\alpha$ such that

(a) if α, β, γ are elements of $\mathrm{Ob}(\square_n^+)$ with $\alpha \to \beta \to \gamma$ in $\square_n^+$, then $\lambda_{\alpha\gamma} = \lambda_{\alpha\beta} \cdot \lambda_{\beta\gamma}$, and

(b) $e_{\alpha\beta}(U_i^{(\beta)}) \subset U_{\lambda_{\alpha\beta}(i)}^{(\alpha)}$ for any $i \in \Lambda_\beta$, where $e_{\alpha\beta} : X_\beta \to X_\alpha$ is a holomorphic map corresponding to an arrow $\alpha \to \beta$ in $\square_n^+$,

(3.1) (2) if we define $V_i^{(\alpha)} := U_i^{(\alpha)} \times M$ for $\alpha \in \mathrm{Ob}(\square_n^+)$ and $i \in \Lambda_\alpha$, then $\mathcal{V}_\alpha : = \{V_i^{(\alpha)}\}_{i \in \Lambda_\alpha}$ is a Stein covering of $\mathcal{X}_\alpha$ for every $\alpha \in \mathrm{Ob}(\square_n^+)$,

(3) $E_{\alpha\beta | V_i^\beta} : V_i^{(\beta)} \to V_{\lambda_{\alpha\beta}(i)}^{(\alpha)}$ is equal to $e_{\alpha\beta | U_i^{(\beta)}} \times \mathrm{id}_M$ for $\alpha \in \mathrm{Ob}(\square_n^+)$ and $i \in \Lambda_\alpha$, where $E_{\alpha\beta} : \mathcal{X}_\beta \to \mathcal{X}_\alpha$ is a holomorphic map over M corresponding to an arrow $\alpha \to \beta$ in $\square_n^+$, and

(4) $\pi_{\alpha | V_i^{(\alpha)}} = \mathrm{Pr}_M : V_i^{(\alpha)} : = U_i^{(\alpha)} \times M \to M$ (projection to M), where $\pi_\alpha : = \pi \cdot a_\alpha$ and $\pi_0 = \pi$.

We take the Cěch resolution $\mathscr{C}^\cdot(\mathcal{V}_\alpha, \Omega^\cdot_{\mathcal{X}_\alpha/M})$ of the complex $\Omega^\cdot_{\mathcal{X}_\alpha/M}$ with respect to the covering $\mathcal{V}_\alpha$ for each $\alpha \in \square_n$. Then the natural homomorphism

$$s(a_{1\cdot *}\Omega^\cdot_{\mathcal{X}\cdot/M})[1] \to s(a_{1\cdot *}\mathrm{tot}\mathscr{C}^\cdot(\mathcal{V}\cdot, \Omega^\cdot_{\mathcal{X}\cdot/M}))[1]$$

is an isomorphism in $D^+(\mathcal{X}, \boldsymbol{C})$. Since $s(a_{1\cdot *}\mathrm{tot}\mathscr{C}^\cdot(\mathcal{V}\cdot, \Omega^\cdot_{\mathcal{X}\cdot/M}))[1]$ is π_*-acyclic, we have

$$R^\ell_{\mathcal{O}}(\pi) \simeq H^\ell(\pi_* s(a_{1\cdot *}\mathrm{tot}\mathscr{C}^\cdot(\mathcal{V}, \Omega^\cdot_{\mathcal{X}\cdot/M}))[1]).$$

By use of this isomorphism, following the arguments of Katz and Oda in [3], we calculate the Gauss-Mannin connection ∇ on $R^\ell_{\mathcal{O}}(\pi)$. From this the Griffiths transversality follows. We should mention that the analytic local triviality assumption on the family $\mathcal{X}\cdot \xrightarrow{a\cdot} \mathcal{X} \xrightarrow{\pi} M$ is necessary so that this procedure can be carried out in our arguments.

§4. Infinitesimal period map. Let $\mathcal{X}\cdot \xrightarrow{a\cdot} \mathcal{X} \xrightarrow{\pi} M$ be an n-cubic $(n \geq 1)$ hyper-equisingular family of complex projective varieties, parametrized by a complex manifold M. For each $\alpha \in \square_n^+$ we denote by $\mathcal{T}_{\mathcal{X}_\alpha/M}$ the sheaf of germs of holomorphic tangent vector fields along fibers on $\mathcal{X}_\alpha (\mathcal{X}_0 := \mathcal{X}$ for $0 := (0, \ldots, 0) \in \square_n^+)$, and by $\mathcal{T}(\mathcal{X}_{/M}, \mathcal{O}_{\mathcal{X}_\alpha})$ the sheaf of germs of $\mathcal{O}_{\mathcal{X}_\alpha}$-valued derivations θ along fibers on $\mathcal{X}$, i.e., $\theta \in \mathcal{T}(\mathcal{X}_{/M}, \mathcal{O}_{\mathcal{X}_\alpha})$ are $\pi^\cdot\mathcal{O}_M$-linear maps $\mathcal{O}_\mathcal{X} \to a_{\alpha *}\mathcal{O}_{\mathcal{X}_\alpha}$ with the property $\theta(ab) = \theta(a)b + a\theta(b)$ for $a, b \in \mathcal{O}_\mathcal{X}$, where $\pi^\cdot\mathcal{O}_M$ is the topological inverse of the structure sheaf of M by the map π. For each $\alpha \in \square_n^+$ we define

$$ta_\alpha : a_{\alpha *}\mathcal{T}_{\mathcal{X}_\alpha/M} \to \mathcal{T}(\mathcal{X}_{/M}, \mathcal{O}_{\mathcal{X}_\alpha}) \text{ and}$$
$$\omega a_\alpha : \mathcal{T}_{\mathcal{X}/M} \to \mathcal{T}(\mathcal{X}_{/M}, \mathcal{O}_{\mathcal{X}_\alpha})$$

by $ta_\alpha(\theta) := \theta a_\alpha^*$ for $\theta \in a_{\alpha *}\mathcal{T}_{\mathcal{X}_\alpha/M}$, $\omega a_\alpha(\varphi) := a_\alpha^* \varphi$ for $\varphi \in \mathcal{T}_{\mathcal{X}/M}$,

where $a_\alpha^* : \mathcal{O}_\mathcal{X} \to a_{\alpha *}\mathcal{O}_{\mathcal{X}_\alpha}$ denotes the pull-back. We define the *sheaf of germs of holomorphic tangent vector fields along fibers* $\mathcal{T}(a.)$ *of an* n*-cubic hyper-equisingular family* $\mathcal{X}\cdot \xrightarrow{a\cdot} \mathcal{X} \xrightarrow{\pi} M$ *of complex projective varieties, parametrized by a complex manifold* M, by

$$\mathcal{T}(a.) := \mathrm{Ker}\{\oplus_{\alpha \in \square_n^+} a_{\alpha *}\mathcal{T}_{\mathcal{X}_\alpha/M} \to \oplus_{\alpha \in \square_n}\mathcal{T}(\mathcal{X}_{/M}, \mathcal{O}_{\mathcal{X}_\alpha}) : (\theta_\alpha) \to (ta_\alpha(\theta_\alpha) - \omega a_\alpha(\theta_0))\}.$$

Now we are going to define the *Kodaira-Spencer map* of a family $\mathcal{X}\cdot \xrightarrow{a\cdot} \mathcal{X} \xrightarrow{\pi} M$ as a map

$$\rho : \mathcal{T}_M \to R^1\pi_*\mathcal{T}(a.),$$

where $\mathcal{T}_M$ denotes the sheaf of germs of holomorphic tangent vector fields on M. We take a point $o \in M$ and put

$$X_\alpha := (\pi \cdot a_\alpha)^{-1}(o) \ (\alpha \in \square_n), \ X := \pi^{-1}(o).$$

By the "analytic local triviality" of a family $\mathcal{X}\cdot \xrightarrow{a\cdot} \mathcal{X} \xrightarrow{\pi} M$, shrinking M sufficiently small around o, we are allowed to assume that there is a special system of Stein coverings $\mathcal{U}_\alpha : = \{U_i^{(\alpha)}\}_{i \in \Lambda_\alpha}$ of X_α $(\alpha \in \square_n^+)$, subject to the requirements in (3.1). We take such a system of Stein coverings of $\mathcal{X}\cdot \xrightarrow{a\cdot} \mathcal{X} \xrightarrow{\pi} M$ and fix it. In the subsequence we will always calculate with respect to this coverings. For each $\alpha \in \square_n^+$ we denote by $C^p(\mathcal{V}_\alpha, \mathcal{T}_{\mathcal{X}_\alpha/M})$ (resp. $Z^p(\mathcal{V}_\alpha, \mathcal{T}_{\mathcal{X}_\alpha/M})$) the p-th Cěch cochains (resp. the p-th Cěch cocycles) with values in the sheaf $\mathcal{T}_{\mathcal{X}_\alpha/M}$ with respect to the Stein covering $\mathcal{V}_\alpha$. We define a subcomplex $C^p(a.)$ of $\oplus_{\alpha \in \square_n^+} C^p(\mathcal{V}_\alpha, \mathcal{T}_{\mathcal{X}_\alpha/M})$ by

$$C^p(a.) :=$$

(4.1) $$\mathit{Ker}\{\oplus_{\alpha \in \square_n^+} C^p(\mathcal{V}_\alpha, \mathcal{T}_{\mathcal{X}_\alpha/M}) \xrightarrow{\oplus_{\alpha \in \square_n}(ta_\alpha - \omega a_\alpha)} \oplus_{\alpha \in \square_n} C^p(\mathcal{V}_0, \mathcal{T}(\mathcal{X}_{/M}, \mathcal{O}_{\mathcal{X}_\alpha}))\}.$$

Let $(t_1, \cdots, t_m)$ and $(x_i^{(\alpha)1}, \cdots, x_i^{(\alpha)n_\alpha})$ $(\alpha \in \square_n^+$, $i \in \Lambda_\alpha$, $n_\alpha := \dim X_\alpha$ for $\alpha \in \square_n$, $n_0 :=$ the local embedding dimension of $X_0 = X)$ be local coordinate systems on M and $U_i^{(\alpha)}$, respectively (for $X_0 = X$ we take a local embedding $X \subset \boldsymbol{C}^{n_0}$ at each point of X and consider the problem modulo $\mathcal{J}(X)$, the ideal sheaf of X in $\mathcal{O}_{\boldsymbol{C}^{n_0}}$). Then $(x_i^{(\alpha)1}, \cdots, x_i^{(\alpha)n_\alpha}, t_1, \cdots, t_m)$ constitutes a local coordinate system in $V_i^{(\alpha)} := U_i^{(\alpha)} \times M$. We denote by

$$\begin{cases} x_i^{(\alpha)\mu} = \varphi_{ij}^{(\alpha)\mu}(x_j^{(\alpha)1}, \cdots, x_j^{(\alpha)n_\alpha}, t_1, \cdots, t_m) \\ \qquad\qquad\qquad\qquad\qquad (1 \le \mu \le n_\alpha) \\ t_\xi = t_\xi \ (1 \le \xi \le m) \end{cases}$$

the transition functions of local coordinate systems in $U_i^{(\alpha)} \cap U_j^{(\alpha)}$ for $i, j \in \Lambda_\alpha$ with $U_i^{(\alpha)} \cap U_j^{(\alpha)} \neq \emptyset$. They satisfy the compatibility conditions:

$$\varphi_{ik}^{(\alpha)\mu}(x_k^{(\alpha)1}, \cdots, x_k^{(\alpha)n_\alpha}, t) = \varphi_{ij}^{(\alpha)\mu}(\varphi_{jk}^{(\alpha)1}(x_k^{(\alpha)1}, \cdots, x_k^{(\alpha)n_\alpha}, t), \cdots, \varphi_{jk}^{(\alpha)n_\alpha}(x_k^{(\alpha)1}, \cdots, x_k^{(\alpha)n_\alpha}, t), t).$$

Hence

$$\frac{\partial \varphi_{ik}^{(\alpha)\mu}}{\partial t_\xi}(x_k^{(\alpha)}, t) = \sum_{\zeta=1}^{n_\alpha} \frac{\partial \varphi_{ik}^{(\alpha)\mu}}{\partial x_j^{(\alpha)\zeta}}(\varphi_{jk}^{(\alpha)}(x_k^{(\alpha)}, t), t) \frac{\partial \varphi_{jk}^{(\alpha)\zeta}}{\partial t_\xi}(x_k^{(\alpha)}, t) + \frac{\partial \varphi_{ij}^{(\alpha)\mu}}{\partial t_\xi}(\varphi_{jk}^{(\alpha)}(x_k^{(\alpha)}, t), t)$$

This implies that if we define

$$\theta_{ik}^\alpha := \sum_{\mu=1}^{n_\alpha} \sum_{\xi=1}^{m} b_\xi(t) \frac{\partial \varphi_{ik}^{(\alpha)\mu}}{\partial t_\xi}(x_k^{(\alpha)}, t)\left(\frac{\partial}{\partial x_i^{(\alpha)\mu}}\right)$$

for $\tau = \sum_{\xi=1}^{m} b_\xi(t)\left(\frac{\partial}{\partial t_\xi}\right) \in \Gamma(M, \mathcal{T}_M)$, then

$$\theta_\alpha := \{\theta_{ik}^\alpha\}_{i,k \in \Lambda_\alpha} \in Z^1(\mathcal{V}_\alpha, \mathcal{T}_{\mathcal{X}_\alpha/M}).$$

On each $V_i^{(\beta)}$ $(i \in \Lambda_\beta)$ we express the holomorphic map $E_{\alpha\beta} : \mathcal{X}_\beta \to \mathcal{X}_\alpha$ corresponding to an arrow $\alpha \to \beta$ in $\square_n$ as

$$\begin{cases} x_{\lambda_{\alpha\beta}(i)}^{(\alpha)\mu} = \overset{i}{e}_{\alpha\beta,\mu}(x_i^{(\beta)1}, \cdots, x_i^{(\beta)n_\beta}) \ (1 \le \mu \le n_\alpha) \\ t_\xi = t_\xi \quad (1 \le \xi \le m) \end{cases}$$

They satisfy the compatibility conditions:

$$\varphi_{ik}^{(\alpha)\mu}(\overset{k}{e}_{\alpha\beta,1}(x_k^{(\beta)}), \cdots, \overset{k}{e}_{\alpha\beta,n_\alpha}(x_k^{(\beta)}), t) = \overset{i}{e}_{\alpha\beta,\mu}(\varphi_{ik}^{(\beta)}(x_k^{(\beta)}, t)) \ (1 \le \mu \le n_\alpha).$$

Hence

$$\frac{\partial \varphi_{ik}^{(\alpha)\mu}}{\partial t_\xi}(\overset{k}{e}_{\alpha\beta,1}(x_k^{(\beta)}), \cdots, \overset{k}{e}_{\alpha\beta,n_\alpha}(x_k^{(\beta)}), t) = \sum_{\zeta=1}^{n_\beta} \frac{\partial \overset{i}{e}_{\alpha\beta,\mu}}{\partial x_i^{(\beta)\zeta}}(\varphi_{ik}^{(\beta)}(x_k^{(\beta)}, t)) \frac{\partial \varphi_{ik}^{(\beta)\zeta}}{\partial t_\xi}(x_k^{(\beta)}, t)$$

This means $dE_{\alpha\beta}(\theta_\beta) = E_{\alpha\beta}^*(\theta_\alpha)$. Hence $\{\theta_\alpha\}_{\alpha \in \square_n} \in Z^1(a.)$, where $Z^1(a.)$ stands for 1-cycles of complex $C^\cdot(a.)$ defined in (4.1). It is fairly easy to check that for each $\alpha \in \square_n$ θ_α in fact defines an element of $C^1(a_\alpha^{-1}(\mathcal{V}_0), \mathcal{T}_{\mathcal{X}_\alpha/M})$, where $a_\alpha^{-1}(\mathcal{V}_0) := \{a_\alpha^{-1}(V_i^{(0)})\}_{i \in \Lambda_0}$, because $a_\alpha : \mathcal{X}_\alpha \to \mathcal{X}$ is a product family over each $V_i^{(0)} \in \mathcal{V}_0 (i \in \Lambda_0)$. Hence $\{\theta_\alpha\}_{\alpha \in \square_n} \in Z^1(\mathcal{V}_0, \mathcal{T}(a.))$. We define $\check{\rho} : \Gamma(M, \mathcal{T}_M) \to H^1(\mathcal{X}, \mathcal{T}(a.))$ by

$$\check{\rho}(\tau) := \{\theta_\alpha\}_{\alpha \in \square_n} \in \check{H}^1(\mathcal{V}_0, \mathcal{T}(a.)) \quad \text{(Cěch cohomology)} \simeq H^1(\mathcal{X}, \mathcal{T}(a.))$$

for $\tau \in \Gamma(M, \mathcal{T}_M)$. We can see that the map $\check{\rho}$ thus defined is independent of the choice of a system of Stein coverings $\{\mathcal{V}_\alpha\}_{\alpha \in \square_n^+}$ of $\mathcal{X}. \xrightarrow{a.} \mathcal{X}$, subject to the requirements in (3.1) as a map to $H^1(\mathcal{X}, \mathcal{T}(a.))$. Localizing the map $\check{\rho}$ at each point of M, we have the map $\rho : \mathcal{T}_M \to R^1\pi_*\mathcal{T}(a.)$.

4.1 Definition. We call the map ρ thus defined the *Kodaira-Spencer map* of an n-cubic hyper-equisingular family $\mathcal{X}. \xrightarrow{a.} \mathcal{X} \xrightarrow{\pi} M$ of complex projective varieties, parametrized by a complex manifold M.

We define

$$\mathrm{Gr}_F^p(\boldsymbol{R}_{\mathcal{O}_M}^\ell(\pi)) := F^p(R_{\mathcal{O}_M}^\ell(\pi)) / F^{p+1}(R_{\mathcal{O}_M}^\ell(\pi)).$$

Then, by Theorem 3.1, (i),

$$\mathrm{Gr}_F^p(\boldsymbol{R}_{\mathcal{O}_M}^\ell(\pi)) \simeq \boldsymbol{R}^{\ell-p}\pi_*(s(a_{1.*}\Omega_{\mathcal{X}./M}^p[1])).$$

By Theorem 3.1, (iii) (the Griffiths transversality), the Gauss-Mannin connection ∇ on $R_{\mathcal{O}}^\ell(\pi)$ induces the following map:

$$\begin{array}{ccc} \mathrm{Gr}_F^p(\nabla) : \mathrm{Gr}_F^p(R_{\mathcal{O}_M}^\ell(\pi)) & \to & \Omega_M^1 \otimes \mathrm{Gr}_F^{p-1}(R_{\mathcal{O}_M}^\ell(\pi)) \\ \downarrow & & \downarrow \\ \boldsymbol{R}^{\ell-p}\pi_*(s(a_{1.*}\Omega_{\mathcal{X}./M}^p)[1]) & \to & \Omega_M^1 \otimes \boldsymbol{R}^{\ell-p+1}\pi_*(s(a_{1.*}\Omega_{\mathcal{X}./M}^{p-1})[1]). \end{array}$$

This map $\mathrm{Gr}_F^p(\nabla)$ is related to the Kodaira-Spencer map ρ as follows:

4.2 Theorem. *The following diagram commutes up to* $(-1)^{p+1}$:

$$\begin{tikzcd} \tau \arrow[r, rightsquigarrow] & \tau \cdot \mathrm{Gr}_F^p(\nabla) \\ \mathcal{T}_M \arrow[u, phantom, "\Cap"] \arrow[r] \arrow[dr, "\rho"'] & \mathrm{Hom}_{\mathcal{O}_M}(\boldsymbol{R}^{\ell-p}\pi_*(s(a_{1.*}\Omega_{\mathcal{X}./M}^p)[1]), \mathbf{R}^{\ell-p+1}\pi_*(s(a_{1.*}\Omega_{\mathcal{X}./M}^{p-1})[1]) \arrow[u, phantom, "\Cap"] \\ & R^1\pi_*\mathcal{T}(a.) \arrow[u, "\substack{\text{coupling with } \rho(\tau) \\ \text{by "contraction"}}"'] \end{tikzcd}$$

where $\tau \cdot \mathrm{Gr}_F^p(\nabla)$ *is defined to be the contraction of* $\mathrm{Gr}_F^p(\nabla)(\cdot)$ *by* τ.

The proof of this theorem is a straightforward calculation in terms of local coordinates.

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