Copula Fields and their Applications

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(Communicated by Kiyosi ITÔ, M. J. A., Dec. 12, 1995)

0. Introduction. The construction of stochastic processes from a family of consistent probability measures can be done by Kolmogorov's extention theorem (see [1]).

But the construction of stochastic processes from a family of nonconsistent probability measures can not always be done.

In this paper we propose the following problems and give the answers.

(P1). For any $T > 0$ and any family of Borel probability measures $\{\rho(t, dx)\}_{0 \le t \le T}$ on R^d, construct a R^d-valued Markov process $\{X(t)\}_{0 \le t \le T}$ on a probability space (Ω, B, P) such that

(0.1) $P(X(t) \in dx) = \rho(t, dx)$ for all $t \in [0, T]$.

(P2). For any $T > 0$, any family of Borel probability measures $\{\rho(t, dx)\}_{0 \le t \le T}$ on R^d, and any Borel probability measure $\mu(dxdy)$ on R^{2d} for which $\int_{y \in R^d} \mu(dxdy) = \rho(0, dx)$ and for which $\int_{x \in R^d} \mu(dxdy) = \rho(T, dy)$, construct a R^d-valued reciprocal process (see [5]) $\{X(t)\}_{0 \le t \le T}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ such that

(0.2) $P(X(t) \in dx) = \rho(t, dx)$ for all $t \in [0, T]$,

(0.3) $P(X(0) \in dx, X(T) \in dy) = \mu(dxdy)$.

Main idea is that of copula in the multivariate analysis (see [2,7,8]). We give the definition of a **copula field**, extending the idea, directly, to the path space.

We also give the applications to the stochastic control. **(P1)** is related to the stochastic quantizations (see [6] and references therein).

1. Copula fields and one dimensional case. In this section we show how to construct a real valued stochastic process from a family of Borel probability measures on $\boldsymbol{R}$, extending directly the idea of copula, to the path space. We also give the definition of the **copula field**. In this section we denote by I the parameter space.

Let us give the definition of a copula for a real valued stochastic process which is well defined from [7], Theorems 6.2.4, 6.2.5.

Definition 1.1. For any real valued stochastic process $\{X(t)\}_{t \in I}$ on a probability space $(\Omega, \boldsymbol{B}, P)$, the family $(C_A^X(u_1, \cdots, u_{\#(A)}))_{A \subset I, \#(A) < \infty}$ of copulas which satisfies the following is called a **copula for** $\{\boldsymbol{X(t)}\}_{\boldsymbol{t \in I}}$; for any $A = \{t_1^A, \cdots, t_{\#(A)}^A\} \subset I$ and any $x_1, \cdots, x_{\#(A)} \in R$

(1.1) $P(X(t_1^A) \le x_1, \cdots, X(t_{\#(A)}^A) \le x_{\#(A)}) =$
$$C_A^X(F_{t_1^A}^X(x_1), \cdots, F_{t_{\#(A)}^A}^X(x_{\#(A)})),$$
where we put $F_t^X(x) = P(X(t) \le x)$.

Before we give the definition of a copulas field for a real valued stochastic process, let us give some notations. Denote by $DF(R)$ the set of all continuous distribution functions on R. For $F \in DF(R)$, we can define the functions $F^*(u)$ $(0 \le u \le 1)$ by the following; put

$$F^*(0) \equiv \begin{cases} \max\{x ; F(x) = 0\} & \text{if } 0 \in Range(F), \\ -\infty & \text{if } 0 \notin Range(F), \end{cases}$$

(1.2) $F^*(u) \equiv \min\{x ; F(x) = u\}$ for $0 < u < 1$,

$$F^*(1) \equiv \begin{cases} \min\{x ; F(x) = 1\} & \text{if } 1 \in Range(F), \\ \infty & \text{if } 1 \notin Range(F) \end{cases}$$

(see [7], p. 49). Put $DF(R)^* \equiv \{F^* ; F \in DF(R)\}$; $DF(R)_I \equiv \{\{F_t\}_{t \in I} ; F_t \in DF(R) (t \in I)\}$; $DF(R)_I^* \equiv \{\{F_t^*\}_{t \in I} ; F_t \in DF(R) (t \in I)\}$.

Definition 1.2. For any real valued stochastic process $\{X(t ; \omega)\}_{t \in I, \omega \in \Omega}$ on a probability space $(\Omega, \boldsymbol{B}, P)$, the **copula field** $\{\boldsymbol{C}^X(\boldsymbol{F}^* ; \omega)(t)\}_{t \in I, \boldsymbol{F}^* \in DF(R)_I^*, \omega \in \Omega}$ **for** $\{X(t ; \omega)\}_{t \in I, \omega \in \Omega}$ is defined as follows; for all $t \in I$, $\boldsymbol{F}^* = \{F_s^*\}_{s \in I} \in DF(R)_I^*$, and $P - a.a.\omega$

(1.3) $\boldsymbol{C}^X(\boldsymbol{F}^* ; \omega)(t) = F_t^*(F_t^X(X(t ; \omega)))$.

When there is no confusion, we simply denote the copula field by $\boldsymbol{C}^X(\boldsymbol{F}^*)(t)$, omitting ω.

Remark 1.1. The copula for a real valued stochastic process $\{X(t)\}_{t \in I}$ is uniquely determined if and only if $F_t^X(x)$ is continuous in $x \in R$ for all $t \in I$. Copula field for a real valued stochastic process is unique. F^* is a quasi-inverse of F (see [7], p. 49), and our choice in (1.2) is convenient as we show in the next proposition whose proof is omitted.

Proposition 1.1. For any $F \in DF(R)$, F^* is strictly increasing, left continuous and has a right hand side limits, and the following holds; for any $u \in (0,1)$, and any $y \in R$,

(1.4) $F^*(u) \leq y$ if and only if $u \leq F(y)$.

The next theorem shows that a copula field for a real valued stochastic process is a path space version of the idea of copula.

Theorem 1.2. For any $\{F_t\}_{t\in I} \in DF(R)_I$, and any real valued stochastic process $\{X(t)\}_{t\in I}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ for which $\{F_t^X\}_{t\in I} \in DF(R)_I$, the stochastic process $\{Y(t) \equiv \boldsymbol{C}^X(\{F_s^*\}_{s\in I})(t)\}_{t\in I}$ satisfies the following; for any $n \geq 1$, $t_1, \cdots, t_n \in I (t_i \neq t_j$ if $i \neq j)$, and $y_1, \cdots, y_n \in R$,

(1.5) $$P(Y(t_1) \leq y_1, \cdots, Y(t_n) \leq y_n) = C^X_{\{t_1,\cdots,t_n\}}(F_{t_1}(y_1), \cdots, F_{t_n}(y_n)).$$

In particular, for any $y \in R$,

(1.6) $P(Y(t) \leq y) = F_t(y)$ for all $t \in I$.

Proof. Since (1.6) is a special case of (1.5) (see [7]), we only prove (1.5).
Since $P(0 < F^X_{t_1}(X(t_1)), \cdots, F^X_{t_n}(X(t_n)) < 1) = 1$, we have

(1.7) $$\begin{aligned} &P(Y(t_1) \leq y_1, \cdots, Y(t_n) \leq y_n) \\ &= P(F^*_{t_1}(F^X_{t_1}(X(t_1))) \leq y_1, \cdots, F^*_{t_n}(F^X_{t_n}(X(t_n))) \leq y_n) \\ &= P(F^X_{t_1}(X(t_1)) \leq F_{t_1}(y_1), \cdots, F^X_{t_n}(X(t_n)) \leq F_{t_n}(y_n)) \text{ (from Proposition 1.1)} \\ &= C^X_{\{t_1,\cdots,t_n\}}(F^X_{t_1}(z_1), \cdots, F^X_{t_n}(z_n)) \\ &= C^X_{\{t_1,\cdots,t_n\}}(F_{t_1}(y_1), \cdots, F_{t_n}(y_n)). \end{aligned}$$

Here we put $z_i = \sup\{x ; F^X_{t_i}(x) \leq F_{t_i}(y_i)\}$ for $1 \leq i \leq n$.

Q.E.D.

We get the following proposition easily.

Proposition 1.3. For any $\boldsymbol{F}^* \in DF(R)^*_I$, and any real valued stochastic process $\{X(t)\}_{t\in I}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ for which $\{F_t^X\}_{t\in I} \in DF(R)_I$, the following holds.

(1). If $\{X(t)\}_{t\in I}$ is a Markov process, then so is $\{\boldsymbol{C}^X(\boldsymbol{F}^*)(t)\}_{t\in I}$.

(2). If $\{X(t)\}_{t\in I}$ is a reciprocal process, then so is $\{\boldsymbol{C}^X(\boldsymbol{F}^*)(t)\}_{t\in I}$.

As an application of Theorem 1.2, let us construct stochastic processes with special time dependence.

Theorem 1.4. For any $T > 0$, any family of distribution functions $\{F_t\}_{t\in[0,T]}$ on R for which $F_t \in DF(R)$ for $0 < t < T$, and any Borel probability measure $\mu(dxdy)$ on R^2 for which $\mu((-\infty, x] \times (-\infty, \infty)) = F_0(x)$ and for which $\mu((-\infty, \infty) \times (-\infty, y]) = F_T(y)$, there exists a real valued reciprocal process $\{Y(t)\}_{0\leq t\leq T}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ such that for all $x, y \in R$

(1.8) $P(Y(t) \leq x) = F_t(x)$ for all $0 \leq t \leq T$,

(1.9) $$P(Y(0) \leq x, Y(T) \leq y) = \mu((-\infty, x] \times (-\infty, y]).$$

Proof. From Theorem 2.1 in [5], and the first part of section 3 in [5], there exists a real valued reciprocal process $\{X(t)\}_{0\leq t\leq T}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ such that for $t \in (0, T)$

(1.10) $$\begin{aligned} &P(X(0) \in dx, X(t) \in dz, X(T) \in dy) \\ &= (T/(2\pi t(T-t)))^{1/2}\exp(|x-y|^2/(2T) \\ &\quad - |x-z|^2/(2t) - |z-y|^2/(2(T-t))) \\ &\quad dz\mu(dxdy). \end{aligned}$$

This is true from the following. Put for $0 \leq s < t < u \leq T$, $x, y, z \in R$,

(1.11) $$q(s, x ; t, y) = (2\pi(t-s))^{-1/2} \exp(-|y-x|^2/(2(t-s))),$$

$$p(s, x ; t, y ; u, z) = q(s, x ; t, y)q(t, y ; u, z)/q(s, x ; u, z).$$

Then $p(s, x ; t, y ; u, z)$ is a reciprocal transition probability density function (see [5], section 3). For $\mu(dxdy)$ and $p(s, x ; t, y ; u, z)$, there exists a reciprocal process $\{X(t)\}_{0\leq t\leq T}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ such that (1.10) holds (see [5], Theorem 2.1).

Putting

(1.12) $$Y(t) \equiv \begin{cases} \boldsymbol{C}^X(\{F_s^*\}_{s\in(0,T)})(t) & \text{if } 0 < t < T, \\ X(t) & \text{if } t = 0 \text{ or } T, \end{cases}$$

the proof is over from Theorem 1.2 and Proposition 1.3.

Q.E.D.

The following Theorem can be obtained in the same way as Theorem 1.4.

Theorem 1.5. For any $T > 0$, any family of distribution functions $\{F_t\}_{t\in[0,T]}$ on R for which $F_t \in DF(R)$ for $0 < t < T$, there exists a real valued Markov process $\{Y(t)\}_{0\leq t\leq T}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ such that

(1.13) $$P(Y(t) \leq x) = F_t(x) \quad \text{for all } x \in R \text{ and } 0 \leq t \leq T$$

Proof. From Theorem 3.2 in [5], there exists a real valued Markov process $\{X(t)\}_{0\leq t\leq T}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ such that (1.10) holds and that

(1.14) $$P(X(0) \in dx) = dF_0(x), \quad P(X(T) \in dx) = dF_T(x).$$

Putting

(1.15) $Y(t) \equiv \begin{cases} C^X(\{F_s^*\}_{s\in(0,T)})(t) & \text{if } 0 < t < T, \\ X(t) & \text{if } t = 0 \text{ or } T, \end{cases}$

the proof is over from Theorem 1.2 and Proposition 1.3.

Q.E.D.

We close this section by giving the application to the stochastic control (see [3]).

Fix $T > 0$ and a probability space $(\Omega, \boldsymbol{B}, P)$. Let $h(t, x) : [0, T] \times R \mapsto R$, $G_1(x) : R \mapsto R$, and $G_2(x, y) : R^2 \mapsto R$ be bounded measurable, and put for a real valued stochastic process $\{X(t)\}_{0\leq t\leq T}$,

(1.16) $J_1(X) \equiv E\Big[\int_0^T k(t, X(t))dt + G_1(X(T))\Big],$

$$J_2(X) \equiv E\Big[\int_0^T k(t, X(t))dt + G_2(X(0), X(T))\Big].$$

The following theorem can be obtained from Theorems 1.4 and 1.5 and the proof is omitted.

Theorem 1.6. (O) For any $A \subset DF(R)_{(0,T)}$ and subset B_1 of the set of all distribution functions on R,

(1.17) $\inf\{J_1(X) ; \{F_t^X\}_{t\in(0,T)} \in A, F_T^X \in B_1\}$
$= \inf\{J_1(X) ; \{F_t^X\}_{t\in(0,T)} \in A, F_T^X \in B_1,$
$\{X(t)\}_{0\leq t\leq T}$ is a Markov process$\}$.

(I) For any $A \subset DF(R)_{(0,T)}$ and subset B_2 of the set of all Borel probability measures on R^2,

(1.18) $\inf\{J_2(X) ; \{F_t^X\}_{t\in(0,T)} \in A, P((X(0), X(T)) \in dxdy) \in B_2\}$
$= \inf\{J_2(X) ; \{F_t^X\}_{t\in(0,T)} \in A, P((X(0), X(T)) \in dxdy) \in B_2,$
$\{X(t)\}_{0\leq t\leq T}$ is a reciprocal process$\}$.

2. Multidimensional case. In this section we consider **(P1)** when $d > 1$ and give the application to the stochastic control theory.

Theorem 2.1. *Let $\{p(t, x)\}_{t\geq 0}$ be a family of probability density functions on R^d. Then there exists a R^d-valued Markov process $\{X(t)\}_{t\geq 0}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ such that*

(2.1)
$$P(X(t) \in dx) = p(t, x)dx \quad \text{for all } t \in [0, \infty).$$

Outline of Proof. Put for $t \geq 0$ and $x_1, \cdots, x_d \in R$,

(2.2) $F_1(t, x_1) = \int_{-\infty}^{x_1} dy_1 \int_{R^{d-1}} p(t, (y_1, y))dy,$

$F_k(t, x_k \mid x_1, \cdots, x_{k-1})$
$$= \int_{-\infty}^{x_k} dy_k \int_{R^{d-k}} p(t, (x_1, \cdots, x_{k-1}, y_k, z)) dz \Big/ \Big(\int_{R^{d-k+1}} p(t, (x_1, \cdots, x_{k-1}, z))dz\Big)$$

if the denominator is positive,
0 otherwise,

for $k = 2, \cdots d$. Then for $t \geq 0$, $k = 2, \cdots d$, and $x_1, \cdots, x_{k-1} \in R$, $F_1(t, x)$ and $F_k(t, x \mid x_1, \cdots, x_{k-1})$ are continuous in x.

For the standard Wiener process $\{W(t)\}_{t\geq 0}$ (see [4]), put for $k = 2, \cdots d$,

(2.3) $X_1(t) = C^{W_1(1+\cdot)}(\{F_1(t, \cdot)^*\}_{t\in[0,\infty)})(t),$

$X_k(t) =$
$$\begin{cases} C^{W_k(1+\cdot)}(\{F_k(\{t, \cdot \mid X_1(t), \cdots, X_{k-1}(t))^*\}_{t\in[0,\infty)})(t) \\ \quad \text{if } \int_{R^{d-k+1}} p(t, (X_1(t), \cdots, X_{k-1}(t), z))dz \neq 0, \\ W_k(1 + t) \qquad \text{otherwise} \end{cases}$$

(see (1.3) for notation). Then it is easy to see that $\{(X_1(t), \cdots, X_d(t))\}_{t\geq 0}$ is a Markov process which satisfies (2.1), inductively in k, since W_i and W_j $(i \neq j)$ are independent of each other, and since $W(\cdot)$ is a Markov process (see [4]).

Q.E.D.

Next we give the application to the stochastic control (see [3]).

Fix $T > 0$ and a probability space $(\Omega, \boldsymbol{B}, P)$. Let $k(t, x) : [0, T] \times R^d \mapsto R$ and $G(x) : R^d \mapsto R$ be bounded measurable, and put for a R^d-valued stochastic process $\{X(t)\}_{0\leq t\leq T}$ on $(\Omega, \boldsymbol{B}, P)$,

(2.4) $J(X) \equiv E\Big[\int_0^T k(t, X(t))dt + G(X(T))\Big].$

The following theorem can be easily obtained from Theorem 2.1, and the proof is omitted.

Theorem 2.2. For any subset A of the set of all familes of $\{F_t\}_{t\in[0,T]}$ of continuous distribution functions on R^d,

(2.5) $\inf\{J(X) ; \{F_t^X\}_{t\in[0,T]} \in A\}$
$= \inf\{J(X) ; \{F_t^X\}_{t\in[0,T]} \in A, \{X(t)\}_{0\leq t\leq T}$ is a Markov process$\}$.

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