On a Construction of the Fundamental Solution for the Free Weyl Equation by Hamiltonian Path-integral

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(Communicated by Kiyosi ITÔ, M. J. A., Jan. 12, 1996)

Key words: Superspace; super Hamilton-Jacobi equation; continuity equation; Feynman; Quantization; spin; non-commutativity.

§1. Introduction and result. Let $\psi(t, q): \boldsymbol{R} \times \boldsymbol{R}^3 \to \boldsymbol{C}^2$ satisfy

$$(1)\ \begin{cases} i\hbar \dfrac{\partial}{\partial t}\psi(t, q) = \boldsymbol{H}\psi(t, q),\ \boldsymbol{H} = -ic\hbar\sigma_j \dfrac{\partial}{\partial q_j}, \\ \psi(0, q) = \underline{\psi}(q). \end{cases}$$

Here $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q))$, the summation w.r.t. $j = 1, 2, 3$ is abbreviated and the Pauli matrices $\{\sigma_j\}$ are, for example represented by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Applying formally the Fourier transformation w.r.t. $q \in \boldsymbol{R}^3$ to (1), we get

$$i\hbar \frac{\partial}{\partial t}\hat{\psi}(t, p) = \hat{\boldsymbol{H}}\hat{\psi}(t, p) \text{ where}$$

$$\hat{\boldsymbol{H}} = c\sigma_j p_j = c\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}.$$

As $\hat{\boldsymbol{H}}^2 = c^2|p|^2\boldsymbol{I}_2$ ($\boldsymbol{I}_2$ stands for 2×2-identity matrix), we easily have

Proposition 1. *For any* $t \in \boldsymbol{R}$,

$$\psi(t, q) = (2\pi\hbar)^{-3/2}\int_{\boldsymbol{R}^3} dp e^{i\hbar^{-1}qp} e^{-i\hbar^{-1}t\hat{\boldsymbol{H}}}\hat{\underline{\psi}}(p) = \int_{\boldsymbol{R}^3} dq' \boldsymbol{E}(t, q, q')\underline{\psi}(q')$$

with

$$\boldsymbol{E}(t, q, q') = (2\pi\hbar)^{-3}\int_{\boldsymbol{R}^3} dp e^{i\hbar^{-1}(q-q')p} \times [\cos(c\hbar^{-1}t|p|)\boldsymbol{I}_2 - ic^{-1}|p|^{-1}\sin(c\hbar^{-1}t|p|)\hat{\boldsymbol{H}}].$$

It seems difficult to imagene from this formula that there exist hidden classical objects for (1).

In spite of this, we claim that there exists the classical mechanics corresponding to the Weyl equation and that a fundamental solution of (1) is constructed as a Fourier integral operator using phase and amplitude functions defined by that classical mechanics. Therefore, the Weyl equation is obtained by quantizing that classical mechanics after Feynman's procedure. Because that Hamiltonian is "of first order both in even and odd variables", we should modify Feynman's argument from Lagrangian to Hamiltonian formulated "path integral".

Main Theorem [Hamilton Path-integral representation].

$$\psi(t, q) = \flat((2\pi\hbar)^{-3/2}\hbar\int_{\mathfrak{R}^{3|2}} d\underline{\xi}d\underline{\pi}\,\mu(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}) \times e^{i\hbar^{-1}\mathcal{S}(t,\bar{x},\bar{\theta},\underline{\xi},\underline{\pi})}\mathcal{F}(\#\underline{\psi})(\underline{\xi}, \underline{\pi}))\Big|_{\bar{x}_{\mathrm{B}}=q}$$

Here, $\mathcal{S}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})$ *and* $\mu(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})$ *are solutions of Hamilton-Jacobi and continuity equations, respectively.*

Remark. We use rather freely the knowledge from superanalysis (= analysis on superspace $\mathfrak{R}^{m|n}$). Roughly speaking, we introduce even and odd variables x_j and θ_k as something-like even and odd forms on "$\boldsymbol{R}^\infty = \Pi_{j=1}^\infty \boldsymbol{R}$", respectively. After introducing the Fréchet-Grassmann structure on $\Lambda\boldsymbol{R}^\infty \sim \mathfrak{R}$, we may develop elementary and real analysis on $\mathfrak{R}^{m|n} = \mathfrak{R}^m_{ev} \times \mathfrak{R}^n_{od} \sim (\Lambda_{ev}\boldsymbol{R}^\infty)^m \times (\Lambda_{od}\boldsymbol{R}^\infty)^n$ as similar as those on $\boldsymbol{R}^m$. In the above, $\mathcal{F}$ denotes the Fourier transformation for functions on superspace $\mathfrak{R}^{3|2}$ and $q = x_{\mathrm{B}} =$ the body part of $x \in \mathfrak{R}^{3|0}$. See, more precisely, [2] or [6].

Detailed proofs will be appeared somewhere [4].

§2. Outline of our procedure. (A) We identify a "spinor" $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q)) : \boldsymbol{R} \times \boldsymbol{R}^3 \to \boldsymbol{C}^2$ with an even supersmooth function $u(t, x, \theta) = u_0(t, x) + u_1(t, x)\theta_1\theta_2 : \boldsymbol{R} \times \mathfrak{R}^{3|2} \to \mathfrak{C}_{ev}$. We denote that identification by

$$\# : L^2(\boldsymbol{R}^3 : \boldsymbol{C}^2) \to \mathcal{L}^2_{SS}(\mathfrak{R}^{3|2}) \text{ and}$$

1991 *Mathematics Subject Classification.* 35A08, 35A30, 35C05, 58D25, 70H99, 81Q60.

$$\flat : \mathcal{L}^2_{SS}(\mathfrak{R}^{3|2}) \to L^2(\boldsymbol{R}^3 : \boldsymbol{C}^2).$$

Here, $\mathfrak{R}^{3|2}$ is the superspace, $u_0(t, x)$, $u_1(t, x)$ are the Grassmann continuation of $\psi_1(t, q)$, $\psi_2(t, q)$, respectively, and

$$\mathcal{L}^2_{SS}(\mathfrak{R}^{3|2}) = \{u(x, \theta) = u_0(x) + u_1(x)\theta_1\theta_2 \mid u_0(q), u_1(q) \in L^2(\boldsymbol{R}^3 : \boldsymbol{C})\}$$

where $x_B = q$.

(B) We represent the matrices $\{\sigma_j\}$ which act on $u(t, x, \theta)$ as follows:

$$\sigma_1\Big(\theta, \frac{\partial}{\partial\theta}\Big) = \theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2},$$
$$\sigma_2\Big(\theta, \frac{\partial}{\partial\theta}\Big) = i\Big(\theta_1\theta_2 + \frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big),$$
$$\sigma_3\Big(\theta, \frac{\partial}{\partial\theta}\Big) = 1 - \theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}.$$

Example. $\sigma_1\Big(\theta, \frac{\partial}{\partial\theta}\Big)(u_0 + u_1\theta_1\theta_2) = u_1 + u_0\theta_1\theta_2$, $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \to \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$.

(C) Therefore, we may correspond the differential operator given by

$$\mathcal{H}\Big(-i\hbar\frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial\theta}\Big) = -ic\hbar\Big(\theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big)\frac{\partial}{\partial x_1}$$
$$(2) \qquad + c\hbar\Big(\theta_1\theta_2 + \frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big)\frac{\partial}{\partial x_2}$$
$$- ic\hbar\Big(1 - \theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}\Big)\frac{\partial}{\partial x_3},$$

which yields the superspace version of the Weyl equation

$$(3) \quad \begin{cases} i\hbar\dfrac{\partial}{\partial t}u(t, x, \theta) \\ \qquad = \mathcal{H}\Big(-i\hbar\dfrac{\partial}{\partial x}, \theta, \dfrac{\partial}{\partial\theta}\Big)u(t, x, \theta), \end{cases}$$
$$u(0, x, \theta) = \underline{u}(x, \theta).$$

Moreover, the "complete Weyl symbol" of (2) is given by

$$(4) \quad \begin{aligned} &\mathcal{H}(\xi, \theta, \pi) \\ &= c(\xi_1 + i\xi_2)\theta_1\theta_2 + c\hbar^{-2}(\xi_1 - i\xi_2)\pi_1\pi_2 \\ &\qquad - ic\hbar^{-1}\xi_3(\theta_1\pi_1 + \theta_2\pi_2). \end{aligned}$$

(D) We consider the classical mechanics corresponding to $\mathcal{H}(\xi, \theta, \pi)$ given by

$$(5) \quad \begin{cases} \dfrac{d}{dt}x_j = \dfrac{\partial\mathcal{H}(\xi, \theta, \pi)}{\partial\xi_j}, \\ \dfrac{d}{dt}\xi_k = -\dfrac{\partial\mathcal{H}(\xi, \theta, \pi)}{\partial x_k} = 0, \\ \dfrac{d}{dt}\theta_l = -\dfrac{\partial\mathcal{H}(\xi, \theta, \pi)}{\partial\xi_l}, \\ \dfrac{d}{dt}\pi_m = -\dfrac{\partial\mathcal{H}(\xi, \theta, \pi)}{\partial x_m} \end{cases}$$

Proposition 2. *There exists a unique global solution $(x(t), \xi(t), \theta(t), \pi(t))$ of (5) with any initial data $(x(0), \xi(0), \theta(0), \pi(0)) = (\underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}) \in \mathfrak{R}^{6|4} = \mathcal{T}^*\mathfrak{R}^{3|2}$. Moreover, for any fixed $(t, \underline{\xi}, \underline{\pi})$, the map defined by*

$$(\underline{x}, \underline{\theta}) \to (\bar{x} = \bar{\theta})$$
$$\bar{x} = x(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \ \bar{\theta} = \theta(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi})$$

gives a supersmooth diffeomorphism from $\mathfrak{R}^{3|2} \to \mathfrak{R}^{3|2}$. Therefore, there exists the inverse map given by

$$(\bar{x}, \bar{\theta}) \to (\underline{x}, \underline{\theta})$$
$$\underline{x} = y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \ \underline{\theta} = w(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}),$$

which satisfies

$$\begin{cases} \bar{x} = x(t, y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\xi}, \omega(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\pi}), \\ \bar{\theta} = \theta(t, y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\xi}, \omega(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\pi}), \\ \underline{x} = y(t, x(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \underline{\xi}, \theta(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \underline{\pi}), \\ \underline{\theta} = \omega(t, x(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \underline{\xi}, \theta(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \underline{\pi}). \end{cases}$$

We put

$$\mathcal{S}_0(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}) = \int_0^t \{\langle\dot{x}(s) \mid \xi(s)\rangle + \langle\dot{\theta}(s) \mid \pi(s)\rangle - \mathcal{H}(x(s), \xi(s), \theta(s), \pi(s))\}ds,$$

and

$$\mathcal{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \langle\underline{x} \mid \underline{\xi}\rangle + \langle\underline{\theta} \mid \underline{\pi}\rangle + \mathcal{S}_0(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi})\Big|_{\substack{\underline{x}=y(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi}) \\ \underline{\theta}=\omega(t,\bar{x},\underline{\xi},\bar{\theta},\underline{\pi})}}.$$

Proposition 3. *$\mathcal{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$ is given by*

$$\begin{aligned} \mathcal{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) &= \langle\bar{x} \mid \underline{\xi}\rangle \\ &+ [|\underline{\xi}|\cos(c\hbar^{-1}t|\underline{\xi}|) - i\underline{\xi}_3\sin(c\hbar^{-1}t|\underline{\xi}|)]^{-1} \\ &\times [|\underline{\xi}|\langle\bar{\theta} \mid \underline{\pi}\rangle - \hbar\sin(c\hbar^{-1}t|\underline{\xi}|)(\underline{\xi}_1 + i\underline{\xi}_2)\bar{\theta}_1\bar{\theta}_2 \\ &\qquad - \hbar^{-1}\sin(c\hbar^{-1}t|\underline{\xi}|)(\underline{\xi}_1 - i\underline{\xi}_2)\underline{\pi}_1\underline{\pi}_2]. \end{aligned}$$

Moreover, it satisfies the following Hamilton-Jacobi equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{S}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) + \mathcal{H}\Big(\dfrac{\partial\mathcal{S}}{\partial\bar{x}}, \bar{\theta}, \dfrac{\partial\mathcal{S}}{\partial\bar{\theta}}\Big) = 0, \\ \mathcal{S}(0, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \langle\bar{x} \mid \underline{\xi}\rangle + \langle\bar{\theta} \mid \underline{\pi}\rangle. \end{cases}$$

Now, we put

$$\mathcal{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \operatorname{sdet}\begin{pmatrix} \dfrac{\partial^2\mathcal{S}}{\partial\bar{x}\partial\underline{\xi}} & \dfrac{\partial^2\mathcal{S}}{\partial\bar{\theta}\partial\underline{\xi}} \\ \dfrac{\partial^2\mathcal{S}}{\partial\bar{x}\partial\underline{\pi}} & \dfrac{\partial^2\mathcal{S}}{\partial\bar{\theta}\partial\underline{\pi}} \end{pmatrix}$$

(sdet = super determinant).

Then, we get

Proposition 4. *$\mathcal{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = |\underline{\xi}|^{-2}[|\underline{\xi}|\cos(c\hbar^{-1}t|\underline{\xi}|) - i\underline{\xi}_3\sin(c\hbar^{-1}t|\underline{\xi}|)]^2$.*

It satisfies the following continuity equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{D} + \dfrac{\partial}{\partial\bar{x}}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\xi}\Big) + \dfrac{\partial}{\partial\bar{\theta}}\Big(\mathcal{D}\dfrac{\partial\mathcal{H}}{\partial\pi}\Big) = 0, \\ \mathcal{D}(0, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = 1. \end{cases}$$

In the above, the argument of $\mathcal{D}$ is $(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$, *those of* $\frac{\partial \mathscr{H}}{\partial \xi}$ *and* $\frac{\partial \mathscr{H}}{\partial \pi}$ *are* $\left(\frac{\partial \mathscr{S}}{\partial \bar{x}}, \bar{\theta}, \frac{\partial \mathscr{S}}{\partial \bar{\theta}}\right)$, *respectively.*

From here, we change the order of variables $\bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi} \in \mathscr{T}^*\mathfrak{R}^{3|2} = \mathfrak{R}^{6|4}$ to $\bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi} \in \mathfrak{R}^{3|2} \times \mathfrak{R}^{3|2}$ (this change corresponds to the process from classical to quantum).

We define an operator

$$(\mathscr{U}(t)\underline{u})(\bar{x}, \bar{\theta}) = (2\pi\hbar)^{-3/2}\hbar \iint_{\mathfrak{R}^{3|2}} d\underline{\xi} d\underline{\pi} \times \mathscr{D}^{1/2}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}) e^{i\hbar^{-1}\mathscr{S}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})} \mathscr{F}\underline{u}(\underline{\xi}, \underline{\pi}).$$

The function $u(t, \bar{x}, \bar{\theta}) = (\mathscr{U}(t)\underline{u})(\bar{x}, \bar{\theta})$ will be shown as a desired solution for (3).

(E) On the other hand, using Fourier transformation, we have readily that

$$\mathscr{H}\left(-i\hbar\frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}\right) = \hat{\mathscr{H}}$$

where $\hat{\mathscr{H}}$ is a (Weyl type) pseudo-differential operator with symbol $\mathscr{H}(\xi, \theta, \pi)$, that is,

$$(\hat{\mathscr{H}}u)(x, \theta) = (2\pi\hbar)^{-3}\hbar \iint d\xi d\pi dy d\omega \, e^{i\hbar^{-1}(\langle x-y|\xi\rangle + \langle \theta-\omega|\pi\rangle)} \mathscr{H}\left(\xi, \frac{\theta+\omega}{2}, \pi\right) u(y, \omega).$$

Theorem 5. (1) *For* $t \in \mathbf{R}$, $\mathscr{U}(t)$ *is well defined unitary operator in* $\mathscr{L}^2_{SS}(\mathfrak{R}^{3|2})$.

(2)(*i*) $\mathbf{R} \ni t \to \mathscr{U}(t) \in \mathbf{B}(\mathscr{L}^2_{SS}(\mathfrak{R}^{3|2}), \mathscr{L}^2_{SS}(\mathfrak{R}^{3|2}))$ *is continuous.*

(*ii*) *For* $\underline{u} \in \mathscr{C}_{SS,0}(\mathfrak{R}^{3|2})$, *we put* $u(t, \bar{x}, \bar{\theta}) = (\mathscr{U}(t)\underline{u})(\bar{x}, \bar{\theta})$. *Then, it satisfies*

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}u(t, \bar{x}, \bar{\theta}) = \hat{\mathscr{H}}u(t, \bar{x}, \bar{\theta}), \\ u(0, \bar{x}, \bar{\theta}) = \underline{u}(\bar{x}, \bar{\theta}). \end{cases}$$

(F) Remarking $\flat\hat{\mathscr{H}} \# \phi = \boldsymbol{H}\phi$ and putting $\mathbf{U}(t)\phi = \flat\mathscr{U}(t) \# \phi$, we have

Theorem 6. (1) *For* $t \in \boldsymbol{R}$, $\mathbf{U}(t)$ *is well defined unitary operator in* $L^2(\boldsymbol{R}^3 : \boldsymbol{C}^2)$.

(2) (*i*) $\boldsymbol{R} \ni t \to \mathbf{U}(t) \in \mathbf{B}(L^2(\boldsymbol{R}^3 : \boldsymbol{C}^2), L^2(\boldsymbol{R}^3 : \boldsymbol{C}^2))$ *is continuous.*

(*ii*) *For* $\underline{\phi} \in L^2(\boldsymbol{R}^3 : \boldsymbol{C}^2)$, *we put* $\phi(t, q) = (\mathbf{U}(t)\underline{\phi})(q)$. *Then, it satisfies*

$$\begin{cases} i\hbar\dfrac{\partial}{\partial t}\phi(t, q) = \boldsymbol{H}\phi(t, q), \\ \phi(0, q) = \underline{\phi}(q). \end{cases}$$

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