Triangles and Elliptic Curves. VII

By Takashi ONO

Department of Mathematics, The Johns Hopkins University, U. S. A.

(Communicated by Shokichi IYANAGA, M. J. A., Feb. 13, 1996)

This is a continuation of series of papers [2] each of which will be referred to as (I), (II), (III), (IV), (V), (VI) in this paper. In (VI) we considered exclusively real triangles $t = (a, b, c)$ and showed that there is a 1-1 correspondence between the classes of similarity of t's and the isomorphism classes of triples $\boldsymbol{E}_t$'s of elliptic curves. In this paper, we pursue the same theme for those objects rational over any subfield k of $\boldsymbol{R}$. This time, we shall introduce a third object (a quartic surface over $\boldsymbol{Q}$) in addition to triangles and elliptic curves to clarify the matter.

§1. $\boldsymbol{Tr}$ and $\boldsymbol{S_+}$. As in (VI), we begin with the set

(1.1) $Tr = \{t = (a, b, c) \in \boldsymbol{R}^3;\ 0 < a < b + c,\ 0 < b < c + a,\ 0 < c < a + b\}.$

For $t, t' \in Tr$, we write $t \sim t'$ if they are similar, i.e., if $t = rt'$ for some $r \in \boldsymbol{R}$. For any subfield $k \subset \boldsymbol{R}$, put

(1.2) $$Tr(k) = Tr \cap k^3.$$

If $t \sim t'$, $t, t' \in Tr(k)$, note that $t = rt'$ with $r \in k$. So we can speak of the embedding $\widetilde{Tr}(k) \subset \widetilde{Tr}$ of quotients in the obvious way.

Next, we consider the set

(1.3) $$S_+ = \{P = (x, y, z) \in \boldsymbol{R}^3;\ x, y, z > 0,\ (xy)^{\frac{1}{2}} + (yz)^{\frac{1}{2}} + (zx)^{\frac{1}{2}} = 1\},$$

where (and hereafter) we assume that $a^{\frac{1}{2}} > 0$ when $a > 0$. On rationalizing the defining relation in (1.3), we have

(1.4) $$S_+ = \{P \in \boldsymbol{R}^3_+;\ 1 > xy + yz + zx,\ (1-xy-yz-zx)^2 - 4(x+y+z)xyz - 8xyz = 0\},$$

where (and hereafter) we put, for $k \subset R$, $k_+ = \{a \in k;\ a > 0\}$.

For $k \subset \boldsymbol{R}$, we put

(1.5) $$S_+(k) = S_+ \cap k^3.$$

Let A, B, C be angles of $t = (a, b, c)$ so that A is between sides b and c; similarly for B, C. Call θ a map: $Tr \to \boldsymbol{R}^3_+$ given by

(1.6) $\theta(t) = (\tan^2(A/2), \tan^2(B/2), \tan^2(C/2)).$

Since θ is defined by angles only, it induces a map $\tilde{\theta} : \widetilde{Tr} \to \boldsymbol{R}^3_+$.

(1.8) **Theorem.** *For any subfield $k \subset \boldsymbol{R}$, the map $\tilde{\theta}$ induces a bijection*:

$$\widetilde{Tr}(k) \rightleftarrows S_+(k).$$

Proof. By abuse of notation, put

(1.9) $f(\alpha) = \tan\alpha$, $\alpha \in I = (0, \pi/2)$.

Note that f is a monotone increasing function with range $(0, +\infty)$ which satisfies the functional equation

(1.10) $f(\alpha) f(\pi/2 - \alpha) = 1$, $\alpha \in I$,

and the (stronger form of) addition formula

(1.11) $$f(\alpha)f(\beta) + f(\beta)f(\gamma) + f(\gamma)f(\alpha) = 1 \Leftrightarrow \alpha + \beta + \gamma = \pi/2.$$

Now let $t = (a, b, c) \in Tr$ and A, B, C be angles of t as above. Putting $\alpha = A/2$, $\beta = B/2$, $\gamma = C/2$ in (1.9), (1.11), we find that the point $\theta(t) = (f(\alpha)^2, f(\beta)^2, f(\gamma)^2)$ belongs to S_+.

It is obvious that $\theta(t) = \theta(t')$ implies $t \sim t'$. Hence the map $\tilde{\theta} : \widetilde{Tr} \to S_+$ is injective. Next, for a subfield $k \subset \boldsymbol{R}$, let $t = (a, b, c) \in Tr(k)$. Then $\cos A = (b^2 + c^2 - a^2)/2bc$ belongs to k and so does $f(\alpha)^2 = (1 - \cos A)/(1 + \cos A)$; similarly for $f(\beta)^2$, $f(\gamma)^2$. Hence $\tilde{\theta}$ induces an injection $\widetilde{Tr}(k) \to S_+(k)$. Finally, it remains to show that this map is surjective. So take any point $P = (x, y, z) \in S_+(k)$. By (1.11), we can find angles A, B, C, $0 < A, B, C < \pi$ so that $A + B + C = \pi$ and that $x = f(\alpha)^2$, $y = f(\beta)^2$, $z = f(\gamma)^2$, where $\alpha = A/2$, etc. Choose a triangle $t = (a, b, c) \in Tr$ with angles A, B, C such that $c = 1$. (In case t happens to be a right triangle, we may assume without loss of generality that $C = \pi/2$, i.e., $c =$ the hypotenuse of $t = 1$.) Note that $\cos A = (1 - f(\alpha)^2)/(1 + f(\alpha)^2) = (1 - x)/(1 + x) \in k$; similarly $\cos B$, $\cos C \in k$. On the other hand, though $\sin A = 2f(\alpha)/(1 + f(\alpha)^2)$ may *not* belong to k in general, note also that $\sin^2 A = 4x/(1 + x)^2 \in k$; similarly for $\sin^2 B$, $\sin^2 C$. On squaring each term of the sine formula, we have

(1.11) $a^2/\sin^2 A = b^2/\sin^2 B = 1/\sin^2 C$,

so we see that a^2, b^2 belong to k. Since $\cos A$, $\cos B$ are both non-zero elements of k (by our assumption on the angle C), the cosine formulas

$a^2 = b^2 + 1 - 2b\cos A$, $b^2 = 1 + a^2 - 2a\cos B$ imply that $t = (a, b, c) \in Tr(k)$ with $\theta(t) = P$. Q.E.D.

§2. *Tr* and $\mathscr{E}$. In (VI) §3, we associated to each $t \in Tr$ an ordered triple $\boldsymbol{E}_t = \{E_a, E_b, E_c)$ of elliptic curves defined over $\boldsymbol{R}$:

$$\begin{aligned} &E_a : Y^2 = x^3 + P_a x^2 + QX, \quad P_a = \frac{1}{2}(b^2 + c^2 - a^2), \\ (2.1)\quad &E_b : Y^2 = x^3 + P_b x^2 + QX, \quad P_b = \frac{1}{2}(c^2 + a^2 - b^2), \\ &E_c : Y^2 = x^3 + P_c x^2 + QX, \quad P_c = \frac{1}{2}(a^2 + b^2 - c^2) \end{aligned}$$

with $Q = -$ (area of $t)^2 = \frac{1}{16}(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2)$.

Let us denote by $\mathscr{E}$ the set of all $\boldsymbol{E}_t$, $t \in Tr$, and call ψ the map $Tr \rightarrow \mathscr{E}$ given by

(2.2) $\qquad \psi(t) = \boldsymbol{E}_t,\ t \in Tr.$

For $t = (a, b, c)$, $t' = (a', b', c') \in Tr$, triples $\boldsymbol{E}_t$, $\boldsymbol{E}_{t'}$ are said to be *isomorphic over* $\boldsymbol{R}$ (written $\boldsymbol{E}_t \cong \boldsymbol{E}_{t'}$) if E_a, E_b, E_c are isomorphic over $\boldsymbol{R}$ to $E_{a'}$, $E_{b'}$, $E_{c'}$, respectively. In this situation, we know that

(2.3)(VI, (3.5)) $\quad t \sim t' \Leftrightarrow \boldsymbol{E}_t \cong \boldsymbol{E}_{t'},\ t, t' \in Tr.$

For a subfield $k \subset \boldsymbol{R}$, if $t = (a, b, c) \in Tr(k)$ then elliptic curves E_a, E_b, E_c are all defined over k. Denote by $\mathscr{E}(k)$ the set of all $\boldsymbol{E}_t$, $t \in Tr(k)$. The map ψ induces a map (written ψ again) $Tr(k) \rightarrow \mathscr{E}(k)$. For $t, t' \in Tr(k)$, the isomorphism $\boldsymbol{E}_t \cong \boldsymbol{E}_{t'}$ over k is defined in the obvious way. Assume now that $t \sim t'$, $t, t' \in Tr(k)$; so $t = rt'$, $r \in k$. Since P_a, Q are forms in $\boldsymbol{Q}[a, b, c]$ of degree 2,4, respectively, we have $P_a(t) = r^2 P_a(t')$, $Q(t) = r^4 Q(t')$. Then the map $(X, Y) \mapsto (r^2X, r^3Y)$ induces an isomorphism $E_a \cong E_{a'}$ over k; similarly for E_b, E_c. Denote by $\tilde{\mathscr{E}}(k)$ the quotient of $\mathscr{E}(k)$ defined by isomorphisms over k. Then the map ψ induces a map $\tilde{\psi} : \widetilde{Tr}(k) \rightarrow \tilde{\mathscr{E}}(k)$ which is surjective by the definition of $\mathscr{E}(k)$.

(2.4)**Theorem.** *For any subfield* $k \subset \boldsymbol{R}$, *the map* $\tilde{\psi}$ *is a bijection*:

$$\widetilde{Tr}(k) \rightleftarrows \tilde{\mathscr{E}}(k).$$

Proof. We only have to prove that the map is injective. So assume that $\boldsymbol{E}_t \cong \boldsymbol{E}_{t'}$ over k, t, $t' \in Tr(k)$. Then the isomorphism is, a fortiori, defined over $\boldsymbol{R}$, and our assertion follows from (2.3), Q.E.D.

$$(2.5)\qquad \begin{array}{ccc} \widetilde{Tr}(k) & \xrightarrow[\sim]{\tilde{\theta}} & S_+(k) \\ & \tilde{\psi}\searrow \qquad \nearrow\tilde{\varphi} & \\ & \tilde{\mathscr{E}}(k) & \end{array}$$

Now that we have two bijections $\tilde{\theta}$, $\tilde{\psi}$, we get the third bijection automatically. However, we prefer to find a bijection $\tilde{\varphi} : \tilde{\mathscr{E}}(k) \rightleftarrows S_+(k)$ so that the diagram (2.5) becomes commutative.

§3. $\mathscr{E}$ and $\boldsymbol{S}_+$. When an elliptic curve of the form

(3.1) $\ Y^2 = X^3 + PX^2 + QX$, $P, Q \in \boldsymbol{R}$, $Q < 0$,

is considered, the quantity λ is handier than the invariant j. It is defined by

(3.2) $\qquad \lambda = N/M < 0$

where M, N are determined by the condition

(3.3) $\ Y^2 = X^3 + PX^2 + QX = X(X+M)(X+N)$, $\quad M > 0$, $N < 0$.

For a subfield $k \subset \boldsymbol{R}$ and $t = (a, b, c) \in Tr(k)$, each member E_a, etc., of the triple $\boldsymbol{E}_t \in \mathscr{E}(k)$ is certainly of type (3.1) and so we can speak of the quantity

(3.4) $\ \lambda_a = \lambda(E_a) = (P_a - bc)/(P_a + bc)$
$= -(1 - \cos A)/(1 + \cos A) = -\tan^2(A/2)$,

similarly for b, c.

Call φ the map $\mathscr{E}(k) \rightarrow k^3$ given by

(3.5) $\qquad \varphi(\boldsymbol{E}_t) = (-\lambda_a, -\lambda_b, -\lambda_c).$

In view of (2.4), (3.4), φ induces a map

(3.6) $\qquad \tilde{\varphi} : \tilde{\mathscr{E}}(k) \rightarrow k^3.$

Furthermore, by (1.6), we find $\tilde{\theta} = \tilde{\varphi}\tilde{\psi}$ and hence we have proved

(3.7) **Theorem.** *The map* $\tilde{\varphi}$ *gives a bijection*:

$$\tilde{\mathscr{E}}(k) \rightleftarrows S_+(k).$$

§4. A special case. For a subfield $k \subset \boldsymbol{R}$, let us define a subset of $Tr(k)$ given by

(4.1) $\ Tr(k)_H = \{t = (a, b, c) \in Tr(k) ; \Delta_t \in k\}$,

where $\Delta_t = \Delta =$ the area of $t = (s(s-a)(s-b)(s-c))^{\frac{1}{2}} = \frac{1}{2}bc\sin A$. Since $\tan(A/2) = \Delta/(s(s-a))$, etc., already belong to k, we can simplify the description of $Tr_H(k)$. We can replace the quartic surface $S_+(k)$ by a quadric surface

(4.2)

$$C_+(k) = \{(u, v, w) \in k_+^{\ 3} ; uv + vw + wu = 1\}.$$

One modifies the diagram (2.5) as follows:

$$(4.3)\qquad \begin{array}{ccc} \widetilde{Tr}_H(k) & \xrightarrow{\tilde{\tau}} & C_+(k) \\ & \searrow \qquad \nearrow & \\ & \tilde{\mathscr{E}}_H(k) & \end{array}$$

where $\tau : Tr_H(k) \rightarrow C_+(k)$ is given by

(4.4) $\tau(t) = (\tan(A/2), \tan(B/2), \tan(C/2))$.

All other notation in (4.3) should be self-explanatory and the proof goes similarly as before.

Examples and comments. When $k = \boldsymbol{Q}$, elements of $Tr_H(\boldsymbol{Q})$ are called "rational triangles" or "Heron triangles" ([1] Chap. V). Heron of Alexandria noted that $t = (13, 14, 15)$ belongs to $Tr_H(\boldsymbol{Q})$ with $\Delta = 84$. By our map (4.4) it corresponds to the point (1/2, 4/7, 2/3) of the quadric $C_+(\boldsymbol{Q})$. On the other hand, by our map (1.6) it corresponds to the point (1/4, 16/49, 4/9) of the quartic $S_+(\boldsymbol{Q})$.

Obviously, every right triangle $t = (a, b, c) \in Tr(k)$ belongs to $Tr_H(k)$. Assume that $C = \pi/2$; hence $a^2 + b^2 = c^2$. Then $\tau(t) = (a/(b+c), b/(a+c), 1)$ and $\theta(t) = (a^2/(b+c)^2, b^2/(a+c)^2, 1)$. In both cases the image of right triangles with $C = \pi/2$ is the intersection of the surface in k_+^3 and the plane $z = 1$ (or $w = 1$).

Needless to say, all equilateral triangles $t = (a, a, a)$, $a \in k_+$, are similar and so they correspond to a single point in the quartic surface. If k does not contain $3^{\frac{1}{2}}$, then $t \notin Tr_H(k)$ because $\Delta_t = (3^{\frac{1}{2}}/4)a$.

§5. An involution. For $t = (a, b, c) \in Tr(k)$, put

(5.1) $t' = (a', b', c')$ with $a' = a(s-a)$,
$b' = b(s-b)$, $c' = c(s-c)$, $s = \dfrac{1}{2}(a+b+c)$.

Then one finds

(5.2) $s' - a' = (s-b)(s-c)$, $s' - b' = (s-c)(s-a)$, $s' - c' = (s-a)(s-b)$,

with $s' = \dfrac{1}{2}(a' + b' + c')$. By (5.1), (5.2), we obtain a map: $Tr(k) \rightarrow Tr(k)$. Furthermore, for the image $t'' = (a'', b'', c'')$ of $t' = (a', b', c')$, we get

(5.3) $a'' = a'(s'-a') = ad$, $b'' = bd$, $c'' = cd$,
with $d = (s-a)(s-b)(s-c)$.

In other words, we have $t'' \sim t$ and so the map $t \mapsto t'$ induces an involution * of $\widetilde{Tr}(k)$. The only fixed point of * is the class of equilateral triangle. By the diagram (2.5), we can transplant * on $S_+(k)$ and $\tilde{\mathscr{E}}(k)$. On the surface $S_+(k)$, the involution $P = (x, y, z) \mapsto P = (x^*, y^*, z^*)$ is determined by the relation:

(5.4) $xx^* = yy^* = zz^* = (xyz)/(x(yz)^{\frac{1}{2}} + y(zx)^{\frac{1}{2}} + z(xy)^{\frac{1}{2}})$.

Example (Heron). Let $k = \boldsymbol{Q}$ and $t = (a, b, c) = (13, 14, 15) \in Tr(\boldsymbol{Q})$. We have $s = 21$, $s - a = 8$, $s - b = 7$, $s - c = 6$, $\Delta = (s(s-a)(s-b)(s-c))^{\frac{1}{2}} = 84$, hence $t \in Tr_H(\boldsymbol{Q})$. Next, by (5.1), we have $t' = (a', b', c') = (104, 98, 90)$, $s' = 146$ and $(\Delta')^2 = 16482816 = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$, which means that $t' \notin Tr_H(\boldsymbol{Q})$; in other words, the involution * of $\widetilde{Tr}(\boldsymbol{Q})$ does not respect the subset $\widetilde{Tr}_H(\boldsymbol{Q})$. Passing to the surface $S_+(\boldsymbol{Q})$, we have

$\theta(t) = (1/2^2, 2^4/7^2, 2^2/3^2)$
$\theta(t)^* = (2^5/73, 7^3/(2\cdot 73), (2\cdot 3^2)/73)$.

As for triples of elliptic curves, denoting by $[P, Q]$ for the curve of type (3.1), we have

$\boldsymbol{E}_t = (E_a, E_b, E_c) = ([126, -84^2], [99, 〃], [70, 〃])$,
$\boldsymbol{E}_t^* = (E_{a'}, E_{b'}, E_{c'}) = ([3444, -2^9 \cdot 3^2 \cdot 7^2 \cdot 73], [4656, 〃], [6160, 〃])$.

References

[1] Dickson, L. E.: History of the Theory of Numbers. vol. 2, Chelsea, New York (1971).

[2] Ono, T.: Triangles and elliptic curves. I ~ VI. Proc. Japan Acad., **70A**, 106–108 (1994): **70A**, 223–225 (1994); **70A**, 311–314 (1994); **71A**, 104–106 (1995); **71A**, 137–139 (1995); **71A**, 184–186 (1995).

I take this opportunity to make a correction to my paper (VI). On p. 186, in (4.6), $x^3 + 4x^2 - 3x$ should read $x^3 + 2x^2 - 3x$.