On the Degrees of Irrationality of Hyperelliptic Surfaces

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1. Let L be a field, which is a finitely generated extension of a ground field k, and assume that $tr.deg._k L = n$. We denote by $d_r(L)$ the degree of irrationality of L over k, which is defined to be the number (cf. [2], [5]):

$$\min\left\{ m \;\middle|\; \begin{array}{l} m = [L : k(x_1, \dots, x_n)], \text{ where } x_1, \dots, \\ x_n \text{ are algebraically independent ele-} \\ \text{ments of } L. \end{array} \right\}$$

We call the field $k(x_1, \dots, x_n)$, which defines the value $d_r(L)$, a maximal rational subfield of L and write *m.r.subf.* for short. For an algebraic variety V defined over k, we define the degree of irrationality of V to be $d_r(k(V))$, where $k(V)$ is the rational function field of V. Clearly it is a birational invariant of algebraic varieties. In other words it is the minimal degree of a dominant rational map from V to the projective n-space. Hence, when $n = 1$, it coincides with the gonality of a curve. In case k is not algebraically closed, for example $k = \boldsymbol{Q}$, we feel a great interest in the value d_r. Because, d_r seems to have some relations with the least number $[k' : k]$ such that the variety V has many rational points over k' (see, e.g. [1]). But it is very difficult to find this value. We assume that $k = \boldsymbol{C}$ hereafter. In this note we announce the results for $d_r(S)$ of hyperelliptic surfaces S. Details will appear elsewhere.

2. Let S denote a hyperelliptic surface. Of course we have that $d_r(S) \geq 2$. First we give examples.

Example 1. Let A_i $(i = 1,2)$ be the abelian surface defined by the following period matrix:

$$\Omega_1 = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \beta \end{pmatrix} \text{ or } \Omega_2 = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 1/2 & \beta \end{pmatrix},$$

where $\Im\alpha \neq 0$ and $\Im\beta \neq 0$. Let g be the automorphism of A_i defined by

$$g(z_1, z_2) = (z_1 + 1/2, -z_2).$$

Then $g^2 = id$ on A and $S_i = A_i/g$ is a hyperelliptic surface. Moreover letting $h(z_1, z_2) = (-z_1, z_2)$, we see that h defines an automorphism of S_i and S_i/h becomes a rational surface. Note that A_i/h and A_i/gh are (birationally equivalent to) a ruled surface with irregularity 1 and a K3 surface respectively (cf. [6]).

Let K_S and $\sim$ denote the canonical divisor of S and the linear equivalence of divisors respectively. Then we have the following

Lemma 2. *Suppose that there is an automorphism φ of S with an order $d(\neq 1)$ such that S/φ is rational. Then $d = 2,3,4$ or 6, and moreover the following facts hold true:*

(1) *If $d = 2$ or 4, then $2K_S \sim 0$.*

(2) *If $d = 3$, then $3K_S \sim 0$.*

(3) *If $d = 6$, then $2K_S \sim 0$ or $3K_S \sim 0$.*

Using this lemma, we obtain the following

Theorem 3. *$d_r(S) = 2$ if and only if $2K_S \sim 0$, i.e., S is isomorphic to one of the surfaces in Example 1.*

Before considering other surfaces, we present some more examples.

Example 4. Let A_i $(i = 1,2)$ be the abelian surface defined by the following period matrix:

$$\Omega_1 = \begin{pmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & 0 & \omega \end{pmatrix} \text{ or}$$

$$\Omega_2 = \begin{pmatrix} 1 & 0 & (\omega - 1)/3 & 0 \\ 0 & 1 & (\omega - 1)/3 & \omega \end{pmatrix},$$

where $\omega = \exp(2\pi\sqrt{-1}/3)$. Let g_i be the automorphism of A_i defined by

$$g_1 z = (z_1 + (\omega + 2)/3, \omega z_2) \text{ and}$$
$$g_2 z = (z_1 + 1/3, \omega z_2),$$

where $z = (z_1, z_2)$. Then $g_i^3 = id$ on A and $S_i = A_i/g_i$ is a hyperelliptic surface. Moreover letting

$$h_1 z = \omega z \text{ and } h_2 z = (\omega z_1, \omega z_2 + 2/3),$$

we see that h_i defines an automorphism of S_i and S_i/h_i becomes a rational surface. Note that A/h, A/gh and A/gh^2 are (birationally equivalent to) a rational surface, a K3 surface and a ruled surface with irregularity 1 respectively, where we put $A = A_i$, $g = g_i$ and $h = h_i$.

These examples are unique in the following sense.

Theorem 5. *For hyperelliptic surfaces S the following conditions (i) and (ii) are equivalent:*

(*i*) $d_r(S) = 3$ *and* $k(S)$ *is a Galois extension of some* $m.r.subf.$

(*ii*) S *is isomorphic to one of the surfaces in Example* 4.

Remark 6. (1) The inequality $d_r(S) \geq 3$ in [5, Theorem 3] for a hyperelliptic surface is an error, which is resulted from dropping the consideration of one possible case in the proof p. 636.

(2) The abelian surfaces A_i in Example 4 are isomorphic to $E_\omega \times E_\omega$, where E_ω is the elliptic curve $\boldsymbol{C}/(1, \omega)$. This abelian surface is the unique one satisfying that $d_r(A) = 3$ and $k(A)$ is a Galois extension of some $m.r.subf.$ (cf. [4]).

Now we consider the remaining surfaces. Let $\mathrm{Bs}\,|D|$ denote the set of the base points of the complete linear system $|D|$. In order to determine d_r, we use the following

Lemma 7. *Suppose that there is a smooth curve* C *of genus* g *on* S *satisfying one of the following conditions* (*i*) *or* (*ii*). *Then* $d_r(S) \leq g$.

(i) $g = 3$, or

(ii) $g \geq 4$ and $\mathrm{Bs}\,|C| = \emptyset$.

Following the table in Suwa [3, Theorem], we classify hyperelliptic surfaces into 7 classes as follows: Let n be the least number satisfying $nK_S \sim 0$. Then $n = 2,3,4$ or 6. Corresponding to these numbers, we say that S is of type I, -II, -III and -IV respectively. Moreover we divide S into a class (i) and -(ii), if the period matrix of A (which is an unramified covering of S of degree n) is represented as a product type or not respectively. Note that if S is of type IV, then it necessarily belongs to the class (i).

Now, we have two fibrations $f_1 : S \to E$ and $f_2 : S \to \boldsymbol{P}^1$, where f_1 [reap. f_2] defines a structure of elliptic fiber bundle over an elliptic curve E [resp. elliptic surface over a rational curve $\boldsymbol{P}^1$](cf. [3]). Let F_i be a general fiber of f_i, and put $r = (F_1, F_2)$, which is the intersection number of F_1 and F_2. Corresponding to types I, II, III and IV, we have that $r = 2,3,4$ and 6 in the class (i), and $r = 4,9$ and 8 in the class (ii) respectively.

Letting mF_2' be a multiple fiber of f_2 in each type, where m is the largest integer in the multiple fibers of f_2, we put $r_0 = (F_1, F_2')$. Then we infer easily that $r_0 = 1$ in the class (i). On the other hand in the class (ii) we have that $r_0 = 3$ and 2, corresponding to type II and III respectively.

Using the divisor $D = F_1 + F_2'$ or $F_1 + 2F_2'$ and considering a general member of the complete linear system $|D|$, and combining the results above, we can conclude the following

Theorem 8. *The degree of irrationality of* S *is given in the following table*:

	(i)	(ii)
I	2	2
II	3	3 or 4
III	3	3
IV	3	

Note 9. We do not know whether the value 4 is taken in case S belongs to type II and class (ii).

References

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