On the Diophantine Equation $x(x+1)=y(y+1)z^2$

By Kenji KASHIHARA

Department of Mathematics, Anan College of Technology

(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1996)

1. In [1] Mihailov investigated the equation:

$$t_x = t_y k_z \tag{1}$$

where t_x, t_y are the triangular numbers associated to x, y, and k_z the quadrangular number z^2, so that (1) means:

$$\frac{1}{2}x(x+1)=\frac{1}{2}y(y+1)z^2. \tag{2}$$

The following theorem was given in [1].

Theorem 1. *For an arbitrary integer f, $(4f(f+1), f, 2(2f+1))$ is a solution of equation* (2).

Mihailov proved by an elementary method this theorem, which does not yield, however, complete solution of the Diophantine equation (2). In this paper we shall give a complete solution to this equation.

2. We may rewrite the equation (2) as

$$(2x+1)^2-1=\{(2y+1)^2-1\}z^2.$$

Put $X=2x+1$, $Y=2y+1$ then this becomes

$$X^2-(Y^2-1)z^2=1. \tag{3}$$

Let Y be an odd integer with $Y\neq\pm1$ and let (X, z) be an arbitrary integer solution of equation (3). Then by the theory of Pell's equation (see [2]), there exist natural numbers T, U such that

$$X+z\sqrt{Y^2-1}=\pm(T+U\sqrt{Y^2-1})^e,\ e\in\mathbf{Z}.$$

Obviously $T+U\sqrt{Y^2-1}=Y+\sqrt{Y^2-1}$ and hence

$$X+z\sqrt{Y^2-1}=\pm(Y+\sqrt{Y^2-1})^e,\ e\in\mathbf{Z}. \tag{4}$$

We note that from equation (3), X is also odd. In case $Y=\pm1$, we have $y=0$ or $y=-1$ and hence $x=0$ or $x=-1$. Therefore equation (4) gives all the solutions of equation (2).

Theorem 2. *Define S by*

$$S=\{(x, y, z)\mid x, y, z\in\mathbf{Z}, X=2x+1, Y=2y+1, X+z\sqrt{Y^2-1}=\pm(Y+\sqrt{Y^2-1})^e, e\in\mathbf{Z}\}$$

Then S coincides with the set of all the solutions of equation (2).

In the following examples we take the sign plus on the right hand side.

Example 1. In case $e=1$, we have $X=Y$, $z=1$ hence $x=y$. Therefore, for an arbitrary integer f, $(f, f, 1)$ is a solution of equation (2).

Example 2. In case $e=2$, we have $X+z\sqrt{Y^2-1}=2Y^2-1+2Y\sqrt{Y^2-1}$, hence $X=2Y^2-1=8y^2+8y+1$ and $z=2Y=2(2y+1)$, that is $x=4y^2+4y$, $z=4y+2$. Therefore, for an arbitrary integer f, $(4f^2+4f, f, 4f+2)$ is a solution of equation (2).

This confirms Theorem 1.

Example 3. In case $e=3$, we have $X+z\sqrt{Y^2-1}=(Y+\sqrt{Y^2-1})^3=4Y^3-3Y+(4Y^2-1)\sqrt{Y^2-1}$, hence $X=4Y^3-3Y$ and $z=4Y^2-1$, that is $x=16y^3+24y^2+9y$, $z=4(2y+1)^2-1$. Therefore, for an arbitrary integer f, $(16f^3+24f^2+9f, f, 16f^2+16f+3)$ is a solution of equation (2).

References

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