On the Problems of Conformal Maps with Quasiconformal Extension

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1. Introduction. Let $f(z)$ be meromorphic and locally univalent in the unit disk $\boldsymbol{D} = \{z : |z| < 1\}$. Then the Schwarzian derivative of $f(z)$ is defined as

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

It is well-known that if $f(z)$ is locally univalent in $\boldsymbol{D}$ and satisfies

$$|S_f(z)| \leqq \frac{2}{(1-|z|^2)^2} \quad (z \varepsilon \boldsymbol{D}),$$

then $f(z)$ is univalent in $\boldsymbol{D}$. Furthermore, if

$$(1) \qquad |S_f(z)| \leqq \frac{2t}{(1-|z|^2)^2} \quad (z \varepsilon \boldsymbol{D})$$

for some $t (0 \leqq t < 1)$, then $f(z)$ has a quasiconformal extension to the plane.

Chuaqui and Osgood [2] have proved that

Theorem A. Let $f(z)$ be analytic in $\boldsymbol{D}$ with $f(0) = 0, f'(0) = 1$, and $f''(0) = 0$. If $f(z)$ satisfies (1) then

$$A(|z|, -t) \leqq |f(z)| \leqq A(|z|, t)$$

and

$$A'(|z|, -t) \leqq |f'(z)| \leqq A'(|z|, t)$$

for $z \varepsilon \boldsymbol{D}$, where A' means the differentiation of A with respect to z, and $A(z, t)$ is defined as

$$(2)\ A(z, t) = \left(\frac{1}{\sqrt{1-t}}\right)\frac{(1+z)^{\sqrt{1-t}} - (1-z)^{\sqrt{1-t}}}{(1+z)^{\sqrt{1-t}} + (1-z)^{\sqrt{1-t}}}.$$

Using Theorem A, they also proved that

Theorem B. If $f(z)$ which is normalized as in Theorem A is analytic in $\boldsymbol{D}$, and satisfies (1), then $f(z)$ has a Hölder continuous extension to $|z| \leqq 1$ with

$$|f(z_1) - f(z_2)| \leqq \frac{4\pi}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}},$$

for all z_1 and z_2 in $\boldsymbol{D}$. The exponent $\sqrt{1-t}$ is sharp.

In Theorem B, although the exponent $\sqrt{1-t}$ is sharp, the Hölder constant $4\pi/\sqrt{1-t}$ is not sharp.

2. Hölder continuous extension. Our first result on Hölder continuous extension is contained in

Theorem 1. Let $f(z)$ be analytic in $\boldsymbol{D}$ with $f(0) = 0, f'(0) = 1$, and $f''(0) = 0$. If $f(z)$ satisfies (1), then $f(z)$ has a Hölder continuous extension to $|z| \leqq 1$ with

$$|f(z_1) - f(z_2)| \leqq \left(\frac{4}{1-\sqrt{1-t}}\right)^{1-\sqrt{1-t}} \frac{1-\sqrt{1-t}+2^{1-\sqrt{1-t}}\sqrt{1-t}}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}}$$

for all z_1 and z_2 in $|z| \leqq 1$. The exponent $\sqrt{1-t}$ is sharp.

Proof. According to Chuaqui and Osgood [2], we have

$$(3) \qquad |f'(z)| \leqq 4\frac{(1+|z|)^{2\nu-1}(1-|z|)^{2\nu-1}}{((1+|z|)^{2\nu} + (1-|z|)^{2\nu})^2} \quad (2\nu = \sqrt{1-t}),$$

and

$$(4) \qquad |f'(z)| \leqq \frac{4^{1-2\nu}}{(1-|z|)^{1-2\nu}}$$

for $z \varepsilon \boldsymbol{D}$. Let z_1 and $z_2 (z_1 \neq z_2)$ be arbitrary points in $\boldsymbol{D}$ and choose $\rho = 1 - (1-2\nu)|z_1 - z_2|/2$. Then, from (4), we have

$$\begin{aligned}|f(z_1) - f(z_2)| &\leqq \left|\int_{z_1}^{\rho z_1} f'(z)dz\right| + \left|\int_{\rho z_1}^{\rho z_2} f'(z)dz\right| + \left|\int_{\rho z_2}^{z_2} f'(z)dz\right| \\
&\leqq 2\int_\rho^1 \frac{4^{1-2\nu}}{(1-r)^{1-2\nu}}dr + \frac{4^{1-2\nu}}{(1-\rho)^{1-2\nu}}|z_1 - z_2| \\
&= \frac{4^{1-\sqrt{1-t}}}{\sqrt{1-t}}\left(\frac{1-\sqrt{1-t}}{2}\right)^{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}} \\
&\quad + \frac{4^{1-\sqrt{1-t}}\,2^{1-\sqrt{1-t}}}{(1-\sqrt{1-t})^{1-\sqrt{1-t}}}|z_1 - z_2|^{\sqrt{1-t}} \\
&= \frac{1}{\sqrt{1-t}}\left(\frac{8}{1-\sqrt{1-t}}\right)^{1-\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}} \\
&\leqq \frac{8}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}} \\
&< \frac{4\pi}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}}.\end{aligned}$$

This gives a better result than Theorem B.

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Moreover, noting that

$$F(r) = \frac{4(1+r)^{2\nu-1}(1-r)^{2\nu-1}}{((1+r)^{2\nu}+(1-r)^{2\nu})^2}$$

is increasing for $0 < r < 1$, we estimate $|f(z_1) - f(z_2)|$ more precisely. Let $t = (1+r)/(1-r)$, $1 - 2\nu = k = 1 - \sqrt{1-t}$, and $\rho = 1 - k|z_1 - z_2|/2$. Then, by (3), we obtain

$$|f(z_1) - f(z_2)| \leqq \left|\int_{z_1}^{\rho z_1} f'(z)dz\right| + \left|\int_{\rho z_1}^{\rho z_2} f'(z)dz\right| + \left|\int_{\rho z_2}^{z_2} f'(z)dz\right|$$

$$\leqq 4\int_{(1+\rho)/(1-\rho)}^{\infty} \frac{t^{k-2}}{(1+t^{k-1})^2}dt + \frac{4((1+\rho)/(1-\rho))^k}{(1+\rho)^{2-k}(1+((1+\rho)/(1-\rho))^{k-1})^2}|z_1 - z_2|$$

$$= \left(\frac{4}{1-k}\right)\frac{(1-\rho)^{1-k}}{(1+\rho)^{1-k}+(1-\rho)^{1-k}} + \frac{4(1-\rho)^{-k}(1+\rho)^{-k}|z_1-z_2|}{((1+\rho)^{1-k}+(1-\rho)^{1-k})^2}$$

$$= \frac{4(2/k)^k|z_1-z_2|^{1-k}}{(2-k|z_1-z_2|/2)^{1-k}+(k|z_1-z_2|/2)^{1-k}}\left(\frac{k}{2(1-k)} + \frac{1}{(2-k|z_1-z_2|/2)^k((2-k|z_1-z_2|/2)^{1-k+}(k|z_1-z_2|/2)^{1-k})}\right)$$

$$\leqq \left(\frac{4}{1-\sqrt{1-t}}\right)^{1-\sqrt{1-t}} \frac{1-\sqrt{1-t}+2^{1-\sqrt{1-t}}\sqrt{1-t}}{\sqrt{1-t}}|z_1 - z_2|^{\sqrt{1-t}}.$$

This completes the proof of Theorem 1.

The following example gives a lower bound for the best Hölder constant M_0.

Example 1. Let $A(z, t)$ be the function defined in (2). Then we have

$$\lim_{\substack{|z|<1\\ z\to 1}} \frac{|A(z,t) - A(1,t)|}{|z-1|^{\sqrt{1-t}}} = \frac{1}{\sqrt{1-t}}\lim_{\substack{|z|<1\\ z\to 1}}\left|\frac{2}{(1+z)^{\sqrt{1-t}}+(1-z)^{\sqrt{1-t}}}\right| = \frac{1}{\sqrt{1-t}}2^{1-\sqrt{1-t}}.$$

Thus the best Hölder constant M_0 must satisfy

$$\frac{2^{1-\sqrt{1-t}}}{\sqrt{1-t}} \leqq M_0 \leqq \left(\frac{4}{1-\sqrt{1-t}}\right)^{1-\sqrt{1-t}} \frac{1-\sqrt{1-t}+2^{1-\sqrt{1-t}}\sqrt{1-t}}{\sqrt{1-t}}.$$

3. Quasiconformal extension. Next we consider conformal mappings that can be extended to quasiconformal mappings. Let $\boldsymbol{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $\boldsymbol{D}$. It is an interesting problem to determine whether a function $f(z) \in \boldsymbol{A}$ is univalent in $\boldsymbol{D}$ or not, and if it is, whether $f(z)$ has a quasiconformal extension on the whole plane $\boldsymbol{C}$. There are many works on this topic. For example, there are Nehari criteria [4], Becker criteria [1], and so on.

Let $f(z) \in \boldsymbol{A}$, and let $g(z)$ be define by

$$g(z) = \frac{f'(x)(1-|x|^2)}{f((z+x)/(1+\bar{x}z)) - f(x)} = \frac{1}{z} + h(z, x).$$

Then $f(z)$ is univalent in $\boldsymbol{D}$ if and only if $g(z)$ is univalent in $\boldsymbol{D}$.

Ozaki and Nunokawa [6] showed that

Theorem C. In order that the function $w = f(z)$ to be univalent in $\boldsymbol{D}$, it is sufficient that

$$|h'(z, x)| \leqq 1 \quad (z \in \boldsymbol{D})$$

for some $x \in \boldsymbol{D}$.

Nunokawa, Obradović and Owa [5] used the corollary of Theorem C to show that

Theorem D. Suppose that $f(z) \in \boldsymbol{A}$, $f(z)/z \neq 0$ for $0 < |z| < 1$, and $|(z/f(z))''| \leqq 1$ $(z \in \boldsymbol{D})$. Then $f(z)$ is univalent in $\boldsymbol{D}$.

Huang [3] further proved that

Theorem E. Let $f(z) = z/(1 - a_2 z + \phi(z)) = z + a_2 z^2 + \ldots \in \boldsymbol{A}$, where $\phi(z)$ is analytic in $\boldsymbol{D}$, $\phi(0) = \phi'(0) = 0$, and
$|\phi(z_1)/z_1 - \phi(z_2)/z_2| \leqq |z_1 - z_2|$ $(z_1 \in \boldsymbol{D}, z_2 \in \boldsymbol{D})$.
Then $f(z)$ is univalent in $\boldsymbol{D}$.

As a corollary of Theorem E, Huang [3] also proved that

Corollary. Suppose that $f(z) \in \boldsymbol{A}$, $f(z)/z \neq 0$ for $0 < |z| < 1$. If $|(z/f(z))''| \leqq 2$ $(z \in \boldsymbol{D})$, then $f(z)$ is univalent in $\boldsymbol{D}$.

The following example shows that the condition in Theorem E,
$|\phi(z_1)/z_1 - \phi(z_2)/z_2| \leqq |z_1 - z_2|$ $(z_1 \in \boldsymbol{D}, z_2 \in \boldsymbol{D})$,
is best for $f(z)$ to be univalent.

Example 2. Let $f_0(z) = z/(1 - tz^3/2)$, $1 < t < 2$. Since $1 < t < 2$, $f_0(z) \in \boldsymbol{A}$. As in Theorem E, we have

$$\phi_0(z) = \frac{t}{2}z^3 \text{ and } \sup_{z\in\boldsymbol{D}}\left|\left(\frac{\phi_0(z)}{z}\right)'\right| = t.$$

This shows that $\sup_{z\in\boldsymbol{D}}|(\phi_0(z)/z)'|$ approaches to

1, as t does. However, $f_0(z)$ is not univalent in $\boldsymbol{D}$. Since

$$f_0(z_1) - f_0(z_2) = \frac{(z_1 - z_2)(1 - tz_1z_2(z_1 + z_2)/2)}{(1 + tz_1^3/2)(1 + tz_2^3/2)},$$

if we set $G(r_1, r_2) = tr_1r_2(r_1 + r_2)/2$, and let $r_2 = r_1^2$, we obtain

$$G(r_1, r_2) = F(r_1) = \frac{(1 + r_1)r_1^4 t}{2}.$$

We see that $F(0) = 0$, $F(1) = t > 1$, by the continuity of $F(r_1)$, there exists a $r_1^*(0 < r_1^* < 1)$ such that $F(r_1^*) = 1$. Thus $f_0(z)$ is not univalent in $\boldsymbol{D}$.

Now, we show that Theorem C is equivalent to Theorem E. If $g(z) = (1 + zh(z, x))/z$, then $1/g(z) = z/(1 + zh(z, x))$. In this case, $\phi(z) = z(h(z, x) - h(0, x))$ and $\phi(0) = \phi'(0) = 0$. If $|h'(z, x)| \leqq 1$, then we have $|(\phi(z)/z)'| \leqq 1$. On the other hand, if $f(z) = z/(1 + a_2z + \phi(z))$ and satisfies the conditions in Theorem E, then $h(z, 0) = a_2 + \phi(z)/z$ and $|h'(z, 0)| \leqq 1$. So Theorem C is equivalent to Theorem E. This result shows that the condition in Theorem C is also best for $f(z)$ to be univalent in $\boldsymbol{D}$.

Considering the quasiconformal extension problem for $f(z) = z/(1 - a_2z + \phi(z))$, we obtain the following explicit result.

Theorem 2. Let $f(z) = z/(1 - a_2z + \phi(z)) = z + a_2z^2 + \ldots \varepsilon \boldsymbol{A}$, where $\phi(z)$ is analytic in $\boldsymbol{D}$, $\phi(0) = \phi'(0) = 0$, and

$$\left|\frac{\phi(z_1)}{z_1} - \frac{\phi(z_2)}{z_2}\right| \leqq k|z_1 - z_2| \quad (z_1 \varepsilon \boldsymbol{D},\ z_2 \varepsilon \boldsymbol{D})$$

for some $k < 1$. Then the mapping $F(z)$ defined by the formula

$$F(z) = \begin{cases} \dfrac{z}{1 - a_2z + \phi(z)} & \text{for } |z| \leqq 1 \\[2ex] \dfrac{z}{1 - a_2z + z\bar{z}\phi(1/\bar{z})} & \text{for } |z| \geqq 1 \end{cases}$$

is a quasiconformal extension of $f(z)$ onto $\hat{\boldsymbol{C}}$ and $|\mu F(z)| = |F_{\bar{z}}/F_z| \leqq k$.

Proof. Note that $\phi(z)$ is analytic in $\hat{\boldsymbol{D}}$ by the condition for $\phi(z)$. It is easy to show that $F(z)$ is sense-preserving local homeomorphism in $\hat{\boldsymbol{C}}$, and because

$$F_z = \frac{1}{(1 - a_2z + |z|^2\phi(1/\bar{z}))^2}$$

and

$$F_{\bar{z}} = \left(\frac{z}{\bar{z}}\right)^2 \frac{\bar{z}^2\phi(1/\bar{z}) - \bar{z}\phi'(1/\bar{z})}{(1 - a_2z + |z|^2\phi(1/\bar{z}))^2}$$

for $|z| \geqq 1$, the complex dilatation of $F(z)$ satisfies

$$|\mu F(z)| = |F_{\bar{z}}/F_z| = |\bar{z}\phi'(1/\bar{z}) - \bar{z}^2\phi(1/\bar{z})| \leqq k$$

in $\boldsymbol{C}$-$\bar{\boldsymbol{D}}$. Thus $F(z)$ is a quasiconformal in $\hat{\boldsymbol{C}}$. The proof of Theorem 2 is finished.

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