Geometric Measure Theory and Manifolds of Nonnegative Ricci Curvature

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Let M be a complete noncompact riemannian manifold of dimension n. The purpose of the present notes is to announce some advances in the following,

Problem. Suppose that the Ricci curvature of M is everywhere positive. Then, is it true that

$$H_{n-1}(M\ ;\boldsymbol{Z}) = \{0\}?$$

The origin of the conjecture goes back to the paper [12] of S.T. Yau who proved,

Theorem. If M has positive Ricci curvature, then

$$H_{n-1}(M\ ;\boldsymbol{R}) = \{0\}.$$

Our results are,

Theorem 1. *Suppose that M has nonnegative Ricci curvature and the order of volume growth greater than* 1. *Then,*

$$H_{n-1}(M\ ;\boldsymbol{Z}) = \{0\}.$$

Theorem 2. *Suppose that M has nonnegative Ricci curvature and bounded diameter growth* (i.e., *the order of diameter growth equal to* 0). *Then, either*

$$H_{n-1}(M\ ;\boldsymbol{Z}) = \{0\}$$

or else a finite covering space $\tilde{M}$, possibly M itself, admits a riemannian splitting

$$\tilde{M} = \bar{M} \times \boldsymbol{R}$$

where $\bar{M}$ is a compact manifold.

Here, by the *order of volume growth,* we mean the order of growth of the function

$$V(p, r) := \mathbf{V}(B_r(p))$$

as $r \to \infty$, while the *order of diameter growth,* refers to the order of growth of the function

$$\mathrm{diam}(p, r) := \max_{S} \max_{x,y\in S} \{d(x, y)\ ; S \text{ is a connected component of } \partial B_r(p)\}$$

as $r \to \infty$ where $B_r(p)$ is the geodesic ball of radius r with center p. After the works of J. Cheeger, M. Gromov, and M. Taylor [3], and U. Abresch and D. Gromoll [1], we know that for a manifold of nonnegative Ricci curvature, we have

$$0 \leq \text{the order of diameter growth} \leq 1$$

and

$$1 \leq \text{the order of volume growth} \leq n.$$

With regards to the conjecture, there is a previously known result by Z. Shen [10] who showed,

Theorem. If a manifold M of positive Ricci curvature satisfies

$$\limsup_{r\to\infty} \frac{\mathrm{diam}(p, r)}{r} < 1$$

for some $p \in M$, then $H_{n-1}(M\ ;\boldsymbol{Z}) = \{0\}$.

Note that in our results, we merely assume *nonnegative* Ricci curvature and not necessarily positive Ricci curvature. Z. Shen and G. Wei [11] showed that under the presence of nonnegative Ricci curvature, if M has *noncollapsing volume* in the sense that

$$\inf_{x\in M} V(x, 1) > 0,$$

then the order of diameter growth is at most 1 less than the order of volume growth. Therefore, from Theorems 1 and 2, follows,

Corollary. *If M has nonnegative Ricci curvature and noncollapsing volume, then the conclusion of Theorem* 2 *holds. In particular, the answer to the Problem is "true" for manifolds of noncollapsing volume.*

It is also known that under the assumption of nonnegative Ricci curvature, we have the relation

$$\text{the order of volume growth} \leq 1 + (n-1)(\text{the order of diameter growth}).$$

Thus, the only case left unsolved in the Problem is where M has

$$\text{the order of volume growth} = \text{the order of diameter growth} = 1.$$

Under a much stronger volume condition, G. Perelman [9] showed that not only the $n-1$st. but all the homology groups of M vanish and consequently M is contractible. See also J. Cheeger and T. Colding [2] for further information of

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the structure of nonnegatively Ricci curved manifolds with volume growth of order $= n$.

We also point out that U. Abresch and D. Gromoll [1] introduced a different notion of the order of diameter growth. For manifolds with nonnegative Ricci curvature, it follows from part of their results that their order of diameter growth is independent of the choice of the basepoint p. Theorem 2 persists with their definition for bounded diameter growth.

For the proofs of Theorems 1 and 2, we use some geometric-measure-theoretic techniques. To explain our results, for $k \in \boldsymbol{Z}_+, k \leq n-1, p \in M, r < R \in \boldsymbol{R}_+$, let us define the family $\mathfrak{T}_p^k(r, R)$ of all k-dimensional minimal locally integral currents $\boldsymbol{T}$ of multiplicity 1 in M with boundary $\partial \boldsymbol{T} \neq \emptyset$ such that

$$\operatorname{spt}\boldsymbol{T} \subset B_R(p),$$
$$\operatorname{spt}\partial\boldsymbol{T} \subset \partial B_R(p), \text{ and}$$
$$\operatorname{spt}\boldsymbol{T} \cap B_r(p) \neq \emptyset.$$

We denote by $\mathbf{M}^k$ the k-dimensional *mass* in M. Put

$$\gamma_p^k(r, R) := \begin{cases} \inf\{\mathbf{M}^k(T) ; T \in \mathfrak{T}_p^k(r, R)\} & \text{if } \mathfrak{T}_p^k(r, R) \neq \emptyset \text{ and} \\ +\infty & \text{if } \mathfrak{T}_p^k(r, R) \neq \emptyset. \end{cases}$$

Consider the following condition for a point $p \in M$,

$$(\mathscr{G}^k) \quad \liminf_{R \to \infty} \gamma_p^k(r, R) = \infty \quad \text{for each fixed } r \in \boldsymbol{R}_+.$$

We have,

Theorem 3. *Suppose that there esists a point p in a complete noncompact manifold M which satisfies Condition $(\mathscr{G}^k)$. Let $\boldsymbol{K}$ be a k-dimensional integral cycle and $\boldsymbol{L}$ be an $n-k$-dimensional integral cycle in M. Assume that the intersection number of their homology classes satisfy*

$$[\boldsymbol{K}] \cdot [\boldsymbol{L}] \neq 0.$$

Then, there exists a k-dimensional homologically area-minimizing integral current $\boldsymbol{T}$ without boundary such that

$$[\boldsymbol{T}] \cdot [\boldsymbol{L}] = [\boldsymbol{K}] \cdot [\boldsymbol{L}].$$

Theorem 3 may be viewed as a weak generalization of the existence theorem for homologically area minimizing currents in compact manifolds to complete noncompact spaces. To prove Theorem 3, we study the relative homology groups of $B_R(p)$ modulo $\partial B_R(p)$.

In order to reduce to the situation of Theorem 3 in the proofs of Theorems 1 and 2, we use the following two lemmas.

Lemma 1. *Let M be a complete manifold of nonnegative Ricci curvature. If M does not split a line and contains an n-dimensional cycle $\boldsymbol{K}$ which is not homologous to 0, then there is a closed curve c with*

$$[\boldsymbol{K}] \cdot [c] \neq 0.$$

Lemma 2. *Suppose that M has nonnegative Ricci curvature and satisfies either of the conditions assumed in Theorem 1 or 2. Then, M satisfies condition $(\mathscr{G}^{n-1})$ at every point $p \in M$.*

Actually, in the proof of Lemma 1, we only use the fact that M has only one end. Now, we recall the following generalization from our [5] of a theorem of A. Kasue [7] and D. Meyer [8].

Theorem 4. *Let M be a complete noncompact manifold of nonnegative Ricci curvature and let $\boldsymbol{T}$ be a compactly supported area minimizing hypersurface in M with no boundary. Then, there are a relatively compact region $\Delta \subset M$, possibly empty, and a compact $n-1$ dimensional manifold N such that $M \sim \Delta$ decomposes in one of the two possible forms*

$$M \sim \Delta = \begin{cases} N \times \boldsymbol{R} & \text{if } \Delta = \emptyset \text{ or} \\ N \times [0, \infty) & \text{if } \Delta = \emptyset. \end{cases}$$

Combining Lemmas 1 and 2 and Theorems 3 and 4, Theorem 1 follows immediately by contradiction as the manifolds of the forms described in the conclusion of Theorem 4 clearly has the volume growth of order 1. To prove Theorem 2, we study the lift $\tilde{M}$ in more detail and rule out the possibility that the said open region $\Delta \neq \emptyset$. Detailed proofs will appear elsewhere [6] where some other applications of Theorem 3 and Lemma 2 and related results will also be discussed.

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