On the Upper Bounds of the Schur Indices of Simple Finite Groups of Lie Type

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The purpose of this note is to announce some results on the upper bounds of the Schur indices of the irreducible characters of the exceptional finite simple groups of Lie type.

1. We need to review the construction of the generalized Gelfand-Graev characters of finite reductive groups (Kawanaka [10]).

Let K be an algebraically closed field of characteristic $p > 0$, and let $\boldsymbol{F}_q$ be the subfield of K with q elements, q being a power of p. Let G be a connected, reductive algebraic group over K, defined over $\boldsymbol{F}_q$, and let $F : G \to G$ be the corresponding Frobenius endomorphism of G. Throughout the note we shall assume that p is not a bad prime for G. Let G^F be the group of F-fixed points of G.

Let T be an F-stable maximal torus of G contained in an F-stable Borel subgroup of G. Let Σ be the root system of G with respect to T. Let $G^{\boldsymbol{C}}$ be a complex reductive connected Lie group with the same root system Σ as G. Then the nilpotent $Ad(G^{\boldsymbol{C}})$-orbits in Lie $G^{\boldsymbol{C}}$ can be parametrized by a set H_Σ of weighted Dynkin diagrams (Dynkin[3]).

Let $L = \mathrm{Lie}\, G$. We fix $h \in (H_\Sigma)^F$. For an integer i, set $\Sigma(i) = \{r \in \Sigma \mid h(r) = i\}$ and $\Sigma(\geqq i) = \{r \in \Sigma \mid h(r) \geqq i\}$. For $i \geqq 1$, set $L(i) = \bigoplus_{r\in\Sigma(i)} u_r$, where u_r's are the root subspaces of L with respect to T. For $r \in \Sigma$, let X_r be the root vector in L coming from a Chevalley basis in characteristic 0, and let $x_r(\ ) : \mathbf{G}_a \to G$ be the one-parameter subgroup of G (associated with r) defined by $x_r(t) = \exp tX_r$, $t \in G_a$; let $U_r = x_r(G_a)$. For $i \geqq 1$, set $U_i = \langle U_r \mid r \in \Sigma(\geqq i)\rangle$. Let $\phi : U_1 \to L(1) + L(2)$ be the map defined by $\phi(\prod_{r\in\Sigma(\geqq 1)} x_r(t_r)) = \sum_{r\in\Sigma(1)\cup\Sigma(2)} t_r X_r$, $t_r \in \mathbf{G}_a$, where the product is taken over the roots in $\Sigma(\geqq 1)$ arranged in some fixed order.

Let $\kappa : L \times L \to K$ be a non-degenerate bilinear mapping such that $\kappa(Ad(g)X, Ad(g)Y) = \kappa(X, Y)$ for $X, Y \in L$, $g \in G$, and that $\kappa(F(X), F(Y)) = \kappa(X, Y)^q$ for $X. Y \in L$. Let $* : L \to L$ be an opposition automorphism of L defined over $\boldsymbol{F}_q$. Let μ be a fixed complex non-trivial additive character of $\boldsymbol{F}_q$.

Recall that $h \in (H_\Sigma)^F$. Then there is an F-stable unipotent conjugacy class c of G such that $c \cap U_2$ is dense in U_2 and invariant under the translations by elements of U_3. Let $u \in (c \cap U_2)^F$. Then we can define a linear character ξ_u of U_2^F by

$$\xi_u(x) = \mu(\kappa(\phi(u)^*, \phi(x))),\ x \in U_2^F.$$

ξ_u can be extended to a linear character $\tilde{\xi_u}$ of a certain subgroup $U_{1.5}{}^F$ of U_1^F such that $(U_1^F : U_{1.5}{}^F) = (U_{1.5}{}^F : U_2^F)$ ([10, pp. 596–597]). We now put

$$\gamma_u = \mathrm{Ind}_{U_{1.5}{}^F}^{G^F}(\tilde{\xi_u}),$$

which we call the generalized Gelfand-Graev character of G^F associated with u. If $\pi : \tilde{G} \to G$ is the simply-connected covering of the derived group of G and H is a subgroup of G^F containing the group $\pi(\tilde{G}^F)$, then we also put

$$\gamma_u^H = \mathrm{Ind}_{U_{1.5}{}^F}^{H}(\tilde{\xi_u}),$$

Theorem 1. (Kawanaka [11, (2.4.1)(iii)]). *Assume that G is an exceptional adjoint simple algebraic group defined over $\boldsymbol{F}_q$ (p being good for G). Then, for any irreducible character χ of G^F, there is a generalized Gelfand-Graev character γ_u of G^F such that $(\gamma_u, \chi)_{G^F} \neq 0$ and is independent of q.*

2. Let us state some results concerning the rationality of the generalized Gelfand-Graev characters γ_u. In the following, ζ_p is a primitive p-th root of unity, α is a generator of $\mathrm{Gal}(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q})$, and k is the quadratic subfield of $\boldsymbol{Q}(\zeta_p)$.

Lemma 2. *Assume that G is a simply-connected, exceptional simple algebraic group defined over $\boldsymbol{F}_q$ (p being good for G). Then, for any $h \in (H_\Sigma)^F$, there is an element t in T^F such that $t^{p-1} = 1$ (possibly $t^{(p-1)/2} = 1$) and that $\xi_u{}^t = \xi_u{}^{\alpha^2}$ ($\xi_u{}^t(x) = \xi_u(txt^{-1})$, $x \in U_2^F$).*

Proposition 3. *Assume that G is an exceptional simple algebraic group defined over $\boldsymbol{F}_q$ (p being good for G), and, $\pi: \tilde{G} \to G$ being the simply-connected covering of the derived group of G, let H be a subgroup of G^F containing the group $\pi(\tilde{G}^F)$. Let v be any finite place of k, and let k_v be the completion of k at v. Then any γ_u^H is realizable ln k_v.*

If χ is an irreducible character of a finite group and if E is a field of characteristic 0, then we denote by $m_E(\chi)$ the Schur index of χ with respect to E.

Corollary 1. *Let the notation and the assumption be as in Proposition 3. Let χ be any irreducible character of H. Let m be the greatest common divisor of $(\gamma_u^H, \chi)_H$ where γ_u runs over all the generalized Gelfand-Graev characters of G^F. Then $m_{\boldsymbol{Q}}(\chi)$ divides $2m$.*

Corollary 2. *Assume that G is an exceptional adjoint simple algebraic group defined over $\boldsymbol{F}_q$ (p being good for G), and let H be the derived group of G^F. Then there exists an upper bound for the Schur indices of the irreducible characters of H independent of q.*

Remark 1. As to the upper bounds of the Schur indices of finite simple groups, we refer Barry [1], Benard [2], Feit [5], Gow [7, 8], Enomoto-Ohmori [4], Ohmori [12], Specht [13].

Remark 2. Let $H = GL_n(\boldsymbol{F}_q)$. Then we can prove that any γ_u is realizable in $\boldsymbol{Q}$. On the other hand, by Kawanaka [9], for any irreducible character χ of H, there is a γ_u such that $(\gamma_u, \chi)_H = 1$. Therefore we have $m_{\boldsymbol{Q}}(\chi) = 1$ for any χ. This is an alternative proof of Zelevinsky-Gow's theorem [14].

Next let $H = {}^3D_4(q^3)$ ($p \neq 2$). Then we can also prove that any γ_u is realizable in $\boldsymbol{Q}$. On the other hand, if χ is any irreducible character of H, then we see that the number m in Corollary 1 is equal to 1 (Geck [6]). Therefore we have $m_{\boldsymbol{Q}}(\chi) = 1$ for any χ. This is an alternativeproof of Barry's theorem [1].

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