A Characterization of Reflexivity

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Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives

$$(x, y)_{i(s)} := \lim_{t\to 0-(+)} (\|y + tx\|^2 - \|y\|^2)/2t.$$

Note that these mappings are well defined on $X \times X$ and the following properties are valid (see also [1] or [2]):

(i) $(x, y)_i = -(-x, y)_s$ if x, y are in X;

(ii) $(x, x)_p = \|x\|^2$ for all x in X;

(iii) $(\alpha x, \beta y)_p = \alpha\beta(x, y)_p$ for all x, y in X and $\alpha\beta \geqq 0$;

(iv) $(\alpha x + y, x)_p = \alpha\|x\|^2 + (y, x)_p$ for all x, y in X and $\alpha \in \boldsymbol{R}$;

(v) $(x + y, z)_p \leqq \|x\|\|z\| + (y, z)_p$ for all x, y, z in X;

(vi) the element x in X is Birkhoff orthogonal over y in X (we denote $x \perp y$), i.e., $\|x + ty\| \geqslant \|x\|$ for all t in $\boldsymbol{R}$ iff $(y, x)_i \leqq 0 \leqq (y, x)_s$;

(vii) the space X is smooth iff $(y, x)_i = (y, x)_s$ for all x, y in X or iff $(\ ,\ )_p$ is linear in the first variable;

where $p = s$ or $p = i$.

We will use the following well known result due to R.C. James [3]

Theorem (James). The Banach space X is reflexive iff for any closed hyperplane H in X containing the null vector there exists an element $u \in X\setminus\{0\}$ so that $u \perp H$.

The following characterization of reflexivity also holds:

Theorem. Let X be a real Banach space. The following statements are equivalent:

(i) X is reflexive;

(ii) For every $F: X \to \boldsymbol{R}$ a continuous convex mapping on X and for any $x_0 \in X$ there exists an element $u_{F,x_0} \in X$ so that the estimation

$$F(x) \geqq F(x_0) + (x - x_0, u_{F,x_0})_i \tag{1}$$

holds for all x in X.

Proof. "(i) $\Rightarrow$ (ii)". Since F is continuous convex on X, F is subdifferentiable on X, i.e., for every $x_0 \in X$ there exists a functional $f_{x_0} \in X^*$ so that

(2) $F(x) - F(x_0) \geqq f_{x_0}(x - x_0)$ for all x in X.

X being reflexive, then, by James' theorem, there is an element $w_{F,x_0} \in X\setminus\{0\}$ such that $w_{F,x_0} \perp \mathrm{Ker}(f_{x_0})$. Since $f_{x_0}(x)w_{F,x_0} - f(w_{F,x_0})x \in \mathrm{Ker}(f_x)$ for all x in X, by the property (vi), we get that

$$(f_{x_0}(x)w_{F,x_0} - f_{x_0}(w_{F,x_0})x, w_{F,x_0})_i \leqq 0$$
$$\leqq (f_{x_0}(x)w_{F,x_0} - f_{x_0}(w_{F,x_0})x, w_{F,x_0})_s$$

for all x in X, which are equivalent, by the above properties of $(\ ,\ )_p$, with

$$(x, u_{F,x_0})_i \leqq f_{x_0}(x) \leqq (x, u_{F,x_0})_s \text{ for all } x \text{ in } X$$

where

$$u_{F,x_0} := f_{x_0}(w_{F,x_0})w_{F,x_0}/\|w_{F,x_0}\|^2.$$

Now, by (2) we obtain the estimation (1).

"(ii) $\Rightarrow$ (i)". Let H be as in James' theorem and $f \in X^*\setminus\{0\}$ with $H = \mathrm{Ker}(f)$. Then, by (ii), for $F = f$ and $x_0 = 0$, there exists an element $u_f \in X$ so that

$$f(x) \geqq (x, u_f)_i \text{ for all } x \text{ in } X.$$

Substituting x by $(-x)$ we also have

$$f(x) \leqq (x, u_f)_s \text{ for all } x \text{ in } X.$$

Now, we observe that $u_f \neq 0$ (because $f \neq 0$) and then

$$(x, u_f)_i \leqq 0 \leqq (x, u_f)_s \text{ for all } x \text{ in } H,$$

i.e., $u_f \perp H$ and by James' theorem we deduce that X is reflexive.

Corollary 1. Let X be a real Banach space. Then X is reflexive iff for every $p: X \to \boldsymbol{R}$ a continuous sublinear functional on X there is an element u_p in X so that

$$p(x) \geqq (x, u_p)_i \text{ for all } x \text{ in } X.$$

Corollary 2. [2]. Let X be a real Banach space. Then X is reflexive iff for every $f \in X^*$ there is an element u_f in X so that

$$(x, u_f)_i \leqq f(x) \leqq (x, u_f)_s \text{ for all } x \text{ in } X.$$

Corollary 3. [2]. Let X be a real Banach space. Then X is smooth and reflexive iff for all $f \in X^*$ there is an element $u_f \in X$ so that

$$f(x) = (x, u_f)_p \text{ for all } x \text{ in } X$$

where $p = s$ or $p = i$.

References

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[3] R. C. James: Reflexivity and the supremum of linear functionals. Israel J. Math., **13**, 298–300 (1972).