Hecke Correspondences and Betti Cohomology Groups for Smooth Compactifications of Hilbert Modular Varieties of Dimension ≤ 3

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Abstract: We consider Betti cohomology groups of smooth toroidal compactifications of Hilbert modular varieties and representations of Hecke correspondences on them. We study absolute values and zeta functions of eigenvalues of those operators for varieties of dimension ≤ 3. Our main results are Theorems $1 \sim 6$ below.

Notations and introduction. K : a totally real algebraic number field, $\mathcal{O}$ = the principal order of K, $g = [K : \boldsymbol{Q}]$, N : a rational integer ≥ 3, $\Gamma(1) = \{\gamma \in \mathrm{GL}_2(\mathcal{O}) \mid \det \gamma$ is totally positive$\}$, $\Gamma(N) = \{\gamma \in \Gamma(1) \mid \gamma \equiv 1 \pmod{N\mathcal{O}}\}$, $\mathfrak{H}^g$: the Cartesian product of g complex upper half planes, $\Gamma(N) \backslash \mathfrak{H}^g$: the Hilbert modular variety. We fix a regular and projective $\Gamma(1)$-admissible family Σ of polyhedral cone decompositions once for all. Note "$\Gamma(1)$-admissible" = "((the totally positive units group of $\mathcal{O}$) $\ltimes$ (the additive group $\mathcal{O}$))-admissible". For a neat congruence subgroup Γ of $\Gamma(1)$, from Σ, one gets the smooth projective toroidal compactification $M_\Gamma = (\Gamma \backslash \mathfrak{H}^g)^\sim$ of $\Gamma \backslash \mathfrak{H}^g$, cf. Ash et al. [1], Hirzebruch [10].

This note may be regarded as continuation of [8] and [9]. In Theorem 2 of [8] and Theorem 8 of [9] we have given sharp estimates for eigenvalues of "Hecke operators" acting on Betti cohomology groups of arbitrary degrees $d \geq 0$ of smoothly compactified Hilbert modular varieties for all the prime ideals $p\mathcal{O}$ of $\mathcal{O}$ with prime numbers $p \nmid N$, also cf. Remark 2 below. We shall study them on middle Betti cohomology groups also for the other prime ideals of $\mathcal{O}$ in Theorems 1 and 5 $(g \leq 3)$. For this we shall extend the method given in [7], cf. Theorems $1 \sim 3$ in §1. In addition, for any $g > 0$ we shall study "Hecke operators" $\{F_n(T(\mathfrak{U}))\}_{(\mathfrak{U},N)=1}$ with $n \geq 0$ defined adelicly and acting on certain direct sum of Betti cohomology groups of the smooth compactifications, cf. Theorem 4 and the explanation before Theorem 6. We shall consider also zeta functions with Euler products attached to arbitrary common eigen-forms for $\{F_n(T(\mathfrak{U}))\}_{(\mathfrak{U},N)=1}$, cf. Theorem 6.

§1. Treatment without adeles. Write $G^+(\mathcal{O})$ = the monoid $\{\gamma \in M_{2,2}(\mathcal{O}) \mid \det \gamma$ is totally positive$\}$, and D = the Hecke ring $\mathrm{HR}(\Gamma(N), G^+(\mathcal{O}))$. Write $\boldsymbol{C}$ = the algebraic correspondence ring of the cycles of codimension g on $M_{\Gamma(N)} \times_{\mathrm{Spec}\,\boldsymbol{C}} M_{\Gamma(N)}$. Let $\alpha \in G^+(\mathcal{O})$. Put $a = N_{\boldsymbol{Q}}^K(\det \alpha)$. Recall [8] and [9]. The complex analytic morphism $\alpha^\vee : \Gamma(a^2N) \backslash \mathfrak{H}^g \to \Gamma(N) \backslash \mathfrak{H}^g$ induced by $\alpha : \mathfrak{H}^g \to \mathfrak{H}^g$ and $\alpha \Gamma(a^2N) \alpha^{-1} \subset \Gamma(N)$, extends to a unique morphism $\varphi \circ \alpha^\sim \circ \varphi_1 : M_{\Gamma(a^2N)} \to M_{\Gamma(N)}$ (see the explanation next to (2) below.) Let can denote the canonical morphism: $M_{\Gamma(a^2N)} \to M_{\Gamma(N)}$ induced by id.: $\mathfrak{H}^g \to \mathfrak{H}^g$ and $\Gamma(a^2N) \subset \Gamma(N)$. Let $\mathfrak{z}(\Gamma(N)\alpha\Gamma(N))$ denote the scheme theoretic image of the morphism (can, $\varphi \circ \alpha^\sim \circ \varphi_1$): $M_{\Gamma(a^2N)} \to M_{\Gamma(N)} \times_{\mathrm{Spec}\,\boldsymbol{C}} M_{\Gamma(N)} = M^2_{\Gamma(N)}$. It is a cycle of codimension g. By [4], [6] and [9], the map $F : D \to \boldsymbol{C}$, given by $\Gamma(N)\alpha\Gamma(N) \mapsto \mathfrak{z}(\Gamma(N)\alpha\Gamma(N))$, is a ring homomorphism. Let $\xi : \boldsymbol{C} \to H^{2g}(M^2_{\Gamma(N)}, \boldsymbol{C})$ denote the ring homomorphism given by $y \mapsto$ the fundamental class of y, cf. [9], §5. By Künneth we have the anti-$\boldsymbol{C}$-algebra isomorphism: $H^{2g}(M^2_{\Gamma(N)}, \boldsymbol{C}) \cong \Pi_{n=0}^{2g} \mathrm{End}_{\boldsymbol{C}} H^n(M_{\Gamma(N)}, \boldsymbol{C})$, $Z \mapsto (\rho_n(Z))_{n=0}^{2g}$, cf. [9], §5. Put $\Psi = \rho_g \circ \xi \circ F$. We get the anti-$\boldsymbol{C}$-algebra homomorphism ψ = (Id.) $\otimes \Psi : \boldsymbol{C} \otimes_{\boldsymbol{Z}} D \to \mathrm{End}_{\boldsymbol{C}} H^g(M_{\Gamma(N)}, \boldsymbol{C})$ by the scalar extension of Ψ from $\boldsymbol{Z}$ to $\boldsymbol{C}$. For any $x \in \boldsymbol{C} \otimes_{\boldsymbol{Z}} D$, we call $\psi(x)$ the Hecke operator of x on $H^g(M_{\Gamma(N)}, \boldsymbol{C})$, cf. [5], [6], [8] and [9]. Our first main result is

Theorem 1. *Assume* $g \leq 3$. *Write* $\Gamma = \Gamma(N)$. *Let r be an integer* > 0, *let* $\{w_i\}_{i=1}^r$ *be complex numbers* $\neq 0$, *and let* $\{\mathcal{A}_i\}_{i=1}^r \subset G^+(\mathcal{O})$. *Let* λ *denote any eigenvalue of* $\psi(\sum_{i=1}^r w_i \cdot \Gamma\mathcal{A}_i\Gamma)$. *We* *obtain*

(1) $$|\lambda| \leq \sum_{i=1}^{r} |w_i| (\deg \Gamma\mathscr{A}_i\Gamma).$$

If there exists any $\beta \in \cup_{i=1}^{r} \Gamma\mathscr{A}_i\Gamma$ *such that both eigenvalues* u *and* v *of* β *are contained in* K *and such that* u/v *is not a unit of* $\mathscr{O}$, *we obtain strict inequality in* (1). *Here* $\deg \Gamma\mathscr{A}_i\Gamma =$ *the number of* $(\Gamma \backslash \Gamma\mathscr{A}_i\Gamma)$ *for each* $i \in [1, r]$, *and* $|\cdot|$ *denotes the ordinary Archimedean absolute value on* $\boldsymbol{C}$.

Remark 1. Assume $g = 3$. Then $H^1(M_{\Gamma(N)}, \boldsymbol{C}) = \{0\}$, cf. Freitag [3]. Hence by the same argument as in [7] we obtain Theorem 1.

Remark 2. Assume $g \geq 4$. Let $\Gamma = \Gamma(N)$. It is an open problem whether one has (1) or not. On the other hand, for every prime number p remaining prime in K with $p \not| N$, we have got already $|\lambda| \leq 2p^{g/2}$ in [9] and [8] where λ is any eigenvalue of $\psi\left(\Gamma\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma\right)$. Note $\left(\deg \Gamma\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma\right) = 1 + p^g = 1 + N_{\boldsymbol{Q}}^{K}(p\mathscr{O})$ for any prime number p remaining prime in K with $p \not| N$; $2p^{g/2} < 1 + p^g$. (In [9] and [8] we have treated $H^d(M_\Gamma, \mathbf{C})$ for any $d \geq 0$.)

Remark 3. Let $\Gamma = \Gamma(N)$. Hirzebruch and Zagier [11] and Geer ("Hilbert modular surfaces," springer, 1988) treated Hecke correspondences on $\Gamma \backslash \mathfrak{H}^2$. They did not consider, however, Hecke correspondences on smooth compactifications of $\Gamma \backslash \mathfrak{H}^2$. Let $\alpha \in G^+(\mathscr{O})$. Let Θ denote a d-closed differential 2-form on $\Gamma \backslash \mathfrak{H}^2$. They define their action of the Hecke rings on $H_{DR}^2(\Gamma \backslash \mathfrak{H}^2, \boldsymbol{C})$ as follows: {For $\Gamma\alpha\Gamma = \cup_{j=1}^{\mu} \Gamma\alpha_j$ (disjoint), de Rham cohomology class of $\Theta \overset{\Gamma\alpha\Gamma}{\mapsto}$ de Rham cohomology class of $\sum_{j=1}^{\mu} \Theta \circ \alpha_j\}$. Let $\omega = y_1^{-2}dx_1 \wedge dy_1 + y_2^{-2}dx_2 \wedge dy_2$ be the 2-form on $\Gamma \backslash \mathfrak{H}^2$ defined below. Hence, (Their action of $\Gamma\alpha\Gamma$) $([\omega]) = [\sum_{j=1}^{\mu} \omega \circ \alpha_j] = [\mu\omega] = (\deg \Gamma\alpha\Gamma)\ [\omega]$ where $[\omega] =$ de Rham cohomology class of $\omega \in H_{DR}^2(\Gamma \backslash \mathfrak{H}^2, \boldsymbol{C})$. On the other hand we have studied the Hecke correspondences $\mathfrak{z}(\Gamma\alpha\Gamma)$ on M_Γ in [8], [9] and this note, and give Theorems 1 ~ 6 here. Notice the strict inequalities in Theorems 1 and 5 in this note. Also cf. Theorem 8 in [9].

Remark 4. On p. 305 of D. Ramakrishnan's paper "Arithmetic of Hilbert-Blumenthal Surfaces" in Canadian Mathematical Society Conference Proceedings, Volume **7**, 285–370 (1987), A.M.S. the author states "it is not possible to find a suitable smooth compactification $\tilde{S}_K$ on which all the Hecke correspondences can be simultaneously made to act." Our results will show that this statement is mistaken. cf. [4], [5], [6], [8] and [9].

In the rest of §1, we assume $g = 2$, give Theorems 2 and 3, and demonstrate Theorem 1 for $g = 2$. Write $\Gamma = \Gamma(N)$. By the Hodge decomposition, $H^2(M_\Gamma, \boldsymbol{C}) = H^{(2,0)}(M_\Gamma) \oplus_{\boldsymbol{C}} H^{(1,1)}(M_\Gamma) \oplus_{\boldsymbol{C}} H^{(0,2)}(M_\Gamma)$. Since $H^{(1,1)}(M_\Gamma)$ does not consist of only the primitive classes, the proof given in [7] does not apply to the present case directly. Write $S_2(\Gamma) =$ the space of the holomorphic Hilbert cusp forms of weight (2,2) on Γ. We have $S_2(\Gamma) \cong H^{(2,0)}(M_\Gamma) = \overline{H^{(0,2)}(M_\Gamma)}$ by $f(z_1, z_2) \mapsto f(z_1, z_2)dz_1 \wedge dz_2$. From [4] we see that the spaces $H^{(2,0)}(M_\Gamma)$, $H^{(1,1)}(M_\Gamma)$ and $H^{(0,2)}(M_\Gamma)$ are $\psi(x)$-invariant for any $x \in \boldsymbol{C} \otimes_{\boldsymbol{Z}} D$, and that the action of the Hecke operators on $H^{(2,0)}(M_\Gamma)$ is compatible with that on $S_2(\Gamma)$. Following Ash et al. [1] and Shimizu [14], now let us introduce certain truncations $\{X_\theta\}_{\theta \gg 0}$ of $\Gamma \backslash \mathfrak{H}^2$. Let t be the number of the inequivalent cusps of $\Gamma \backslash \mathfrak{H}^2$, and let $\{c_1, c_2, \cdots, c_t\}$ denote a complete set of the Γ-inequivalent cusps. Let θ denote a sufficiently large positive real number. Write $\mathscr{W}_\theta = \{(z_1, z_2) \in \mathfrak{H}^2 \mid (\operatorname{Im} z_1)(\operatorname{Im} z_2) \geq \theta\}$. For each $\nu \in [1, t]$ choose an element $\sigma_\nu \in \{g \in \mathrm{GL}_2(K) \mid \det g$ is totally positive$\}$ with $\sigma_\nu c_\nu = (\sqrt{-1}\,\infty, \sqrt{-1}\,\infty)$. Put $U_\nu(\theta) = \sigma_\nu^{-1}(\mathscr{W}_\theta)$, and $D_\nu(\theta) = \Gamma_{c_\nu} \backslash U_\nu(\theta)$ for each $\nu \in [1, t]$. Here $\Gamma_{c_\nu} =$ the isotropy group of cusp c_ν in Γ. For each $\nu \in [1, t]$ write $D_\nu(\theta)^{\mathrm{int}}$ for the interior of $D_\nu(\theta)$. Write $X_\theta =$ the complement of $(\cup_{\nu=1}^{t} D_\nu(\theta)^{\mathrm{int}})$ in $\Gamma \backslash \mathfrak{H}^2$. Let $(z_1, z_2) \in \mathfrak{H}^2$. Write $z_i = x_i + \sqrt{-1}\, y_i$ $(i = 1, 2)$ with real variables $\{x_i, y_i\}_{i=1}^2$. Write $\omega = y_1^{-2}dx_1 \wedge dy_1 + y_2^{-2}dx_2 \wedge dy_2$, which is a 2-form on $\Gamma \backslash \mathfrak{H}^2$. By Propositions 1.1 and 1.2 in Mumford [13], the functional $\langle\omega\rangle$: {C^∞ differential 2-forms on $M_\Gamma\} \to \boldsymbol{C}$, given by $\tau \mapsto \lim_{\theta \to +\infty} \int_{X_\theta} \omega \wedge \tau$, is a d-closed current on M_Γ. The cohomology class of $(2\pi)^{-1}\langle\omega\rangle \in H^2(M_\Gamma, \boldsymbol{Q})$. We can choose and fix a real analytic Hodge metric Ω_0 on the complex projective manifold M_Γ. For a current $\mathscr{Y}$ on M_Γ, let $\mathscr{Y} = \boldsymbol{H}\mathscr{Y} + d\delta\boldsymbol{G}\mathscr{Y} + \delta d\boldsymbol{G}\mathscr{Y}$ be the orthogonal decomposition of $\mathscr{Y}$ in Potential Theory with respect to Ω_0. We have $\delta d\boldsymbol{G}\langle\omega\rangle = 0$. By C. L. Siegel, $\int_{\Gamma \backslash \mathfrak{H}^2} \omega \wedge \omega > 0$.

By the Hodge index theorem, the signature of the intersection form on $H^{(1,1)}(M_\Gamma)$ is $(1, m-1)$ with $m = \dim_{\boldsymbol{C}} H^{(1,1)}(M_\Gamma)$. For differential 2-forms u_1 and u_2 on M_Γ, write $[u_1, u_2] = \int_{M_\Gamma} u_1 \wedge \overline{u_2}$. We have

(2) There is a $\boldsymbol{C}$-basis $\{\boldsymbol{H}\langle\omega\rangle\} \cup \{\eta_k\}_{k=1}^{m-1}$ of $H^{(1,1)}(M_\Gamma)$ such that (i) $[\boldsymbol{H}\langle\omega\rangle, \boldsymbol{H}\langle\omega\rangle] > 0$; (ii) $[\boldsymbol{H}\langle\omega\rangle, \eta_k] = 0$ for any $k \in [1, m-1]$; (iii) $[\eta_i, \eta_j] = 0$ if $i \neq j$; (iv) $[\eta_k, \eta_k] = -1$ for any $k \in [1, m-1]$.

Let $\alpha \in G^+(\mathcal{O})$. There is a system $\{\alpha_j\}_{j=1}^{\mu}$ of representatives for $\Gamma\backslash\Gamma\alpha\Gamma$ such that $\Gamma\alpha\Gamma = \cup_{j=1}^{\mu} \Gamma\alpha_j$ (disjoint) $= \cup_{j=1}^{\mu} \alpha_j\Gamma$ (disjoint), cf. [15]. Put $a = N_{\boldsymbol{Q}}^K(\det \alpha)$. Then $(\alpha^{-1}\Gamma(N)\alpha \cap \alpha\Gamma(N)\alpha^{-1}) \supset \Gamma(aN)$, cf. [9]. Hence, $\Gamma(a^2N) \subset \alpha^{-1}\Gamma(aN)\alpha \subset \Gamma(N)$ and $\Gamma(a^2N) \subset \alpha\Gamma(aN)\alpha^{-1} \subset \Gamma(N)$. Let φ_1 (resp. φ_2) denote the canonical morphism: $M_{\Gamma(a^2N)} \to (\alpha^{-1}\Gamma(aN)\alpha\backslash\mathfrak{H}^2)^\sim$ (resp. $M_{\Gamma(a^2N)} \to (\alpha\Gamma(aN)\alpha^{-1}\backslash\mathfrak{H}^2)^\sim$), and let φ denote the canonical morphism: $M_{\Gamma(aN)} \to M_{\Gamma(N)}$. The map α (resp. α^{-1}) induces a unique morphism $\alpha^\sim : (\alpha^{-1}\Gamma(aN)\alpha\backslash\mathfrak{H}^2)^\sim \to M_{\Gamma(aN)}$ (resp. $\alpha^{-1\sim}$: $(\alpha\Gamma(aN)\alpha^{-1}\backslash\mathfrak{H}^2)^\sim \to M_{\Gamma(aN)}$). For any differential form $\mathcal{U}$ on $M_{\Gamma(N)}$, we write $\mathcal{U}\circ\alpha$ (resp. $\mathcal{U}\circ\alpha^{-1}$) = the pull back of $\mathcal{U}$ by the morphism $\varphi\circ\alpha^\sim\circ\varphi_1$ (resp. $\varphi\circ\alpha^{-1\sim}\circ\varphi_2$) for short. They are differential forms on $M_{\Gamma(a^2N)}$. Therefore we may regard $\sum_{j=1}^{\mu}\mathcal{U}\circ\alpha_j$ and $\sum_{j=1}^{\mu}\mathcal{U}\circ\alpha_j^{-1}$ on $M_{\Gamma(a^2N)}$ as differential forms on $M_{\Gamma(N)}$. Then we obtain the following two theorems.

Theorem 2. *Notations being as in* (2), *the $\boldsymbol{C}$-linear subspace $\boldsymbol{C}(\boldsymbol{H}\langle\omega\rangle)$ spanned by $\boldsymbol{H}\langle\omega\rangle$ is $\psi(x)$-invariant for any $x \in \boldsymbol{C}\otimes_{\boldsymbol{Z}} D$.* ($D = \mathrm{HR}(\Gamma, G^+(\mathcal{O}))$.)

Theorem 3. *Notations being as in* (2), *the $\boldsymbol{C}$-linear subspace $\sum_{k=1}^{m-1}\boldsymbol{C}\eta_k$ spanned by $\{\eta_k\}_{k=1}^{m-1}$ is $\psi(x)$-invariant for any $x \in \boldsymbol{C}\otimes_{\boldsymbol{Z}} D$.* ($D = \mathrm{HR}(\Gamma, G^+(\mathcal{O}))$.)

For each $i \in [1, r]$ let $\{\mathcal{A}_{i,j}\}_{j=1}^{s(i)}$ be a system of representatives for $\Gamma\backslash\Gamma\mathcal{A}_i\Gamma$. Write $\mathfrak{a}_i = N_{\boldsymbol{Q}}^K(\det \mathcal{A}_{i,j})$ for each $i \in [1, r]$, and $\mathfrak{a} = \Pi_{i=1}^r \mathfrak{a}_i$. For a differential form $\mathcal{U}$ on M_Γ let $\mathcal{U}\circ\mathcal{A}_{i,j}$ (resp. $\mathcal{U}\circ\mathcal{A}_{i,j}^{-1}$) denote the form on $M_{\Gamma(\mathfrak{a}_i^2N)}$ defined above. For each (i, j) let $[\mathcal{U}\circ\mathcal{A}_{i,j}]$ (resp. $[\mathcal{U}\circ\mathcal{A}_{i,j}^{-1}]$) denote the form on $M_{\Gamma(\mathfrak{a}^2N)}$ which is the pullback of $\mathcal{U}\circ\mathcal{A}_{i,j}$ (resp. $\mathcal{U}\circ\mathcal{A}_{i,j}^{-1}$) under the canonical morphism $M_{\Gamma(\mathfrak{a}^2N)} \to M_{\Gamma(\mathfrak{a}_i^2N)}$ by $\mathfrak{a}_i^2N \mid \mathfrak{a}^2N$. We can choose a real analytic Hodge metric Ω_2 on $M_{\Gamma(\mathfrak{a}^2N)}$, and consider the orthogonal decomposition $\boldsymbol{H}_2Y + d\delta_2\boldsymbol{G}_2Y + \delta_2 d\boldsymbol{G}_2Y$ of a current Y on $M_{\Gamma(\mathfrak{a}^2N)}$ with respect to Ω_2 in Potential Theory. Let λ be any eigenvalue of $\psi(\sum_{i=1}^r w_i\cdot\Gamma\mathcal{A}_i\Gamma)$ on $H^{(1,1)}(M_\Gamma)$. By (2) and Theorems 2 and 3 we have only to consider the following two cases.

Case 1 of $\psi(\sum_{i=1}^r w_i\cdot\Gamma\mathcal{A}_i\Gamma)\,\boldsymbol{H}\langle\omega\rangle = \lambda\boldsymbol{H}\langle\omega\rangle$: Let V_1 denote the vector space $\sum_{i=1}^r\sum_{j=1}^{s(i)}\boldsymbol{C}[(\boldsymbol{H}\langle\omega\rangle)\circ\mathcal{A}_{i,j}]$. Define a Hermitian scalar product $(\ ,\ )$: $V_1 \times V_1 \to \boldsymbol{C}$ by $(\mathcal{U}_1, \mathcal{U}_2) = \dfrac{1}{\mathrm{vol}(\Gamma(\mathfrak{a}^2N)\backslash\mathfrak{H}^2)}\int_{M_{\Gamma(\mathfrak{a}^2N)}}\mathcal{U}_1\wedge\overline{\mathcal{U}_2}$ for all $\mathcal{U}_1$ and $\mathcal{U}_2$ in V_1. We can apply the argument in [7]. We obtain that this $(\ ,\ )$ is positive definite, and Case 1 of Theorem 1.

Case 2 of $\psi(\sum_{i=1}^r w_i\cdot\Gamma\mathcal{A}_i\Gamma)\eta = \lambda\eta$ for some non-zero $\eta \in \sum_{k=1}^{m-1}\boldsymbol{C}\eta_k$: Let V_2 denote the vector space $\sum_{k=1}^{m-1}\sum_{i=1}^r\sum_{j=1}^{s(i)}\boldsymbol{C}[\eta_k\circ\mathcal{A}_{i,j}]$. Define a Hermitian scalar product $(\ ,\ )$: $V_2 \times V_2 \to \boldsymbol{C}$ by $(\mathcal{U}_1, \mathcal{U}_2) = \dfrac{-1}{\mathrm{vol}(\Gamma(\mathfrak{a}^2N)\backslash\mathfrak{H}^2)}\int_{M_{\Gamma(\mathfrak{a}^2N)}}\mathcal{U}_1\wedge\overline{\mathcal{U}_2}$ for all $\mathcal{U}_1$ and $\mathcal{U}_2$ in V_2. We can apply the argument in [7]. We obtain that this $(\ ,\ )$ is also positive definite, and Case 2 of Theorem 1.

We may leave the details to the reader. (There is a matrix $P \in G^+(\mathcal{O})$ such that $P^{-1}\beta P$ is diagonal. Put $\mathfrak{b} = N_{\boldsymbol{Q}}^K(\det P)$. Consider $M_{\Gamma(\mathfrak{b}^2N)}$ and M_Γ.)

Using the Petersson scalar product on $S_2(\Gamma)$ we get Theorem 1 also for any common eigen-form in $H^{(2,0)}(M_\Gamma)$ and $H^{(0,2)}(M_\Gamma)$ by the same argument. (For $g = 1$, cf. Drinfeld [2].)

§2. Treatment with adeles. Write $G_{\infty+} = \mathrm{GL}_2^+(\boldsymbol{R})^g$, $\mathfrak{P}_\infty$ = the product of the g infinite primes of K, and h = the number of ideal classes modulo $\mathfrak{P}_\infty$. Take h elements $t_1, \cdots, t_h$ of $K_{\boldsymbol{A}}^\times$ so that $(t_j)_\infty = 1$ and $t_1\mathcal{O}, \cdots, t_h\mathcal{O}$ form a set of representatives for such ideal classes, and put $x_j = \begin{pmatrix}1 & 0\\ 0 & t_j\end{pmatrix}$ for each $j \in [1, h]$. Let $\mathfrak{p}$ denote a maximal ideal of $\mathcal{O}$. Let $\mathcal{M}$ and N be rational integers ≥ 3 with $(\mathcal{M}(\Pi_{j=1}^h t_j)) \mid N$. Write $Y_\mathfrak{p} = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(K_\mathfrak{p}) \mid a-1 \equiv b \equiv c \equiv 0 \pmod{N\mathcal{O}_\mathfrak{p}},\ d \in \mathcal{O}_\mathfrak{p}\right\}$; $W_\mathfrak{p} = \{x \in Y_\mathfrak{p} \mid (\det x) \equiv 1 \pmod{N\mathcal{O}_\mathfrak{p}}\}$; $W = G_{\infty+}\times\Pi_\mathfrak{p} W_\mathfrak{p}$; $Y = \mathrm{GL}_2(K_{\boldsymbol{A}}) \cap (G_{\infty+}\times\Pi_\mathfrak{p} Y_\mathfrak{p})$. (Notice the distinction between our definitions and Shimura's [16], of W and Y.) Write $\Gamma_j = x_jWx_j^{-1} \cap \mathrm{GL}_2(K)$ for each j. Then for each j, Γ_j is neat, and $\Gamma(N^2) \subset \Gamma_j \subset \Gamma(\mathcal{M})$. Let $\mathrm{HR}(W,$

Y) denote the Hecke ring for the monoid $Y \supset$ the group W. Now let us fix any $y \in Y$. Then for each j one can define $\mu(j)$ by the condition that $(\det y) t_j t_{\mu(j)}^{-1} \mathcal{O}$ is the principal class modulo $\mathfrak{P}_\infty$. There is some $\alpha_j \in x_j Y x_{\mu(j)}^{-1} \cap \mathrm{GL}_2(K)$ with $WyW = W x_j^{-1} \alpha_j x_{\mu(j)} W$. Put $a_j = N_{\boldsymbol{Q}}^K(\det \alpha_j)$. Now write Φ_1, Φ and $P_{\mu(j)}$ for the canonical morphisms $M_{\Gamma(a_j^2 N^2)} \to M_{\alpha_j^{-1}\Gamma(a_j N^2)\alpha_j}$, $M_{\Gamma(a_j N^2)} \to M_{\Gamma_j}$ and $M_{\Gamma(a_j^2 N^2)} \to M_{\Gamma_{\mu(j)}}$ respectively, induced by the inclusions of the subgroups. Write $\alpha_j^{\sim}$ for the morphism $M_{\alpha_j^{-1}\Gamma(a_j N^2)\alpha_j} \to M_{\Gamma(a_j N^2)}$ induced by $\alpha_j : \mathfrak{H}^g \to \mathfrak{H}^g$. Write $\mathfrak{z}(\Gamma_j \alpha_j \Gamma_{\mu(j)})$ for the scheme-theoretic image of the morphism $(P_{\mu(j)}, \Phi \circ \alpha_j^{\sim} \circ \Phi_1) : M_{\Gamma(a_j^2 N^2)} \to M_{\Gamma_{\mu(j)}} \times_{\mathrm{Spec}\, \boldsymbol{C}} M_{\Gamma_j}$. It is a cycle of codimension g. By Künneth, $\mathfrak{z}(\Gamma_j \alpha_j \Gamma_{\mu(j)})$ induces a unique element of $\oplus_{n=0}^{2g} \mathrm{Hom}_{\boldsymbol{C}} (H^n(M_{\Gamma_j}, \boldsymbol{C}), H^n(M_{\Gamma_{\mu(j)}}, \boldsymbol{C}))$, for which we write $(\rho_n(\mathfrak{z}(\Gamma_j \alpha_j \Gamma_{\mu(j)})))_{n=0}^{2g}$. Hence WyW induces $F_n(WyW) \stackrel{\text{def}}{=} \sum_{j=1}^h \rho_n(\mathfrak{z}(\Gamma_j \alpha_j \Gamma_{\mu(j)})) \in \mathrm{End}_{\boldsymbol{C}} (\oplus_{j=1}^h H^n(M_{\Gamma_j}, \boldsymbol{C}))$ for each $n \in [0, 2g]$. Extend the domain of F_n to the whole $\mathrm{HR}(W, Y)$ linearly. By the method given in [4], [6] and [9] we obtain:

Theorem 4. *For each* $n \in [0, 2g]$, *the map* $F_n : \mathrm{HR}(W, Y) \to \mathrm{End}_{\boldsymbol{C}} (\oplus_{j=1}^h H^n(M_{\Gamma_j}, \boldsymbol{C}))$ *is an anti-ring homomorphism.*

For a maximal ideal $\mathfrak{p}$ of $\mathcal{O}$ with $\mathfrak{p} \nmid N$, choose any $\pi_\mathfrak{p} \in K_\mathfrak{p}$ with $\mathrm{ord}_\mathfrak{p} \pi_\mathfrak{p} = 1$. Put $T(\mathfrak{p}) = W\begin{pmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix} W$ and $S(\mathfrak{p}) = W\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix} W$. Case of $g = 2$ {For each $j \in [1, h]$, put $\mathcal{V}_0(j) =$ the subspace $\boldsymbol{C}\langle\langle\omega\rangle\rangle$ of $H^2(M_{\Gamma_j}, \boldsymbol{C})$, where $\langle\langle\omega\rangle\rangle$ is the cohomology class of $\langle\omega\rangle$ obtained by replacing M_Γ by M_{Γ_j} in §1. Put $\mathcal{V}_1(j) =$ the orthogonal complement of $\mathcal{V}_0(j)$ with respect to the intersection form: $H^2(M_{\Gamma_j}, \boldsymbol{C}) \times H^2(M_{\Gamma_j}, \boldsymbol{C}) \to \boldsymbol{C}$. As in Theorems 2 and 3 in §1, both $\mathcal{V}_0 = \oplus_{j=1}^h \mathcal{V}_0(j)$ and $\mathcal{V}_1 = \oplus_{j=1}^h \mathcal{V}_1(j)$ are $\mathrm{HR}(W, Y)$-invariant subspaces of $\oplus_{j=1}^h H^2(M_{\Gamma_j}, \boldsymbol{C})$}. There is such an integer $n > 0$ as $\mathfrak{p}^n$ is a principal ideal modulo $\mathfrak{P}_\infty$. Using Theorem 1 in §1 we obtain

Theorem 5. *Assume* $g = 2$ (*resp.* 3). *Take any* $i \in \{0, 1\}$. *Recall* $F_2 : \mathrm{HR}(W, Y) \to \mathrm{End}_{\boldsymbol{C}} (\oplus_{j=1}^h H^2(M_{\Gamma_j}, \boldsymbol{C}))$ (*resp.* $F_3 : \mathrm{HR}(W, Y) \to \mathrm{End}_{\boldsymbol{C}} (\oplus_{j=1}^h H^3(M_{\Gamma_j}, \boldsymbol{C}))$). *Any eigenvalue* $\lambda_\mathfrak{p}$ *of* $F_2(T(\mathfrak{p})) \mid \mathcal{V}_i$ (*resp.* $F_3(T(\mathfrak{p}))$) *satisfies* $|\lambda_\mathfrak{p}| < 1 + N_{\boldsymbol{Q}}^K \mathfrak{p} = \deg W\begin{pmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix} W$ (*strictly smaller*) *for any maximal ideal* $\mathfrak{p}$ *of* $\mathcal{O}$ *with* $\mathfrak{p} \nmid N$.

According to Shimura [16], for each integral ideal $\mathfrak{U}$ of $\mathcal{O}$, let $T(\mathfrak{U})$ denote the element of $\mathrm{HR}(W, Y)$ which is the sum of all different WyW with $y \in Y$ such that $(\det y)\mathcal{O} = \mathfrak{U}$; define $S_N(\mathfrak{U}) \in \mathrm{HR}(W, Y)$ to be $\prod_{\mathfrak{p} : \mathfrak{p} | \mathfrak{U},\ \text{maximal}} S(\mathfrak{p})^{(\mathrm{ord}_\mathfrak{p} \mathfrak{U})}$ if $(\mathfrak{U}, N) = 1$. Define $S_N(\mathfrak{U}) = 0$ if $(\mathfrak{U}, N) \neq 1$. Note that $\mathcal{L} \stackrel{\text{def}}{=} \{T(\mathfrak{p}) \mid \mathfrak{p} \nmid N,\ \mathfrak{p}$ is maximal$\} \cup \{S_N(\mathfrak{p}) \mid \mathfrak{p} \nmid N,\ \mathfrak{p}$ is maximal$\}$ are commutative with one another. Hence, there is a common eigen-form $\boldsymbol{f} \neq \boldsymbol{0}$ of all $\{F_2(\ell)\}_{\ell \in \mathcal{L}}$ in $\mathcal{V}_i$ for each $i \in \{0, 1\}$ if $g = 2$; there is a common eigen-form $\boldsymbol{f} \neq \boldsymbol{0}$ of all $\{F_3(\ell)\}_{\ell \in \mathcal{L}}$ in $\oplus_{j=1}^h H^3(M_{\Gamma_j}, \boldsymbol{C})$ if $g = 3$. Let $\boldsymbol{f}$ be any such common eigen-form for $g \leq 3$. Recall $g = [K : \boldsymbol{Q}]$. We have $(F_g(T(\mathfrak{U})))\boldsymbol{f} = c_\mathfrak{U} \boldsymbol{f}$ for any integral ideal $\mathfrak{U}$ of $\mathcal{O}$ with $(\mathfrak{U}, N) = 1$. Then there is a unique Abelian character χ of the ideal classes modulo $\mathfrak{P}_\infty$ such that $(F_g(S_N(\mathfrak{p})))(\boldsymbol{f}) = \chi(\mathfrak{p})\boldsymbol{f}$ for any maximal ideal $\mathfrak{p}$ of $\mathcal{O}$ with $\mathfrak{p} \nmid N$. Note $|\chi(\mathfrak{p})| = 1$ for any such $\mathfrak{p}$. One has the formal Euler product equation $\sum_{\mathfrak{U} : (\mathfrak{U}, N) = 1} T(\mathfrak{U}) (N_{\boldsymbol{Q}}^K \mathfrak{U})^{-s} = \prod_{\mathfrak{p} : \mathfrak{p} \nmid N,\ \text{maximal}} (1 - T(\mathfrak{p}) (N_{\boldsymbol{Q}}^K \mathfrak{p})^{-s} + S_N(\mathfrak{p}) (N_{\boldsymbol{Q}}^K \mathfrak{p})^{1-2s})^{-1}$. Hence, by Theorem 4, we have the following formal Dirichlet series equation attached to $\boldsymbol{f}$:

$$\sum_{\mathfrak{U} : (\mathfrak{U}, N) = 1} c_\mathfrak{U} (N_{\boldsymbol{Q}}^K \mathfrak{U})^{-s} \tag{6.1}$$

$$= \prod_{\mathfrak{p} : \mathfrak{p} \nmid N,\ \text{maximal}} (1 - c_\mathfrak{p} (N_{\boldsymbol{Q}}^K \mathfrak{p})^{-s} + \chi(\mathfrak{p}) (N_{\boldsymbol{Q}}^K \mathfrak{p})^{1-2s})^{-1}.$$

From Theorem 5 we obtain

Theorem 6. *Assume* $g \leq 3$. *The Euler product in the right side of* (6.1) *converges absolutely and uniformly on any compact subset of* $\{s \in \boldsymbol{C} \mid \mathrm{Re}\ s > 2\}$. (*Hence it becomes a holomorphic function at least on* $\{s \in \boldsymbol{C} \mid \mathrm{Re}\ s > 2\}$.)

Question 1. For every such common eigen-form $\boldsymbol{f}$ as in Theorem 6, does Dirichlet series (6.1) extend holomorphically to the whole of $\boldsymbol{C}$?

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