On an Algebra of a Certain Class of Operators in a Slab Domain in $\mathbf{R}^2$

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Abstract: In this paper a Dirichlet problem for the Laplacian in a domain with a corner in $\boldsymbol{R}^2$ is treated and a new method of construction of a local parametrix for that Dirichlet problem at a corner point is given. In order to construct a parametrix a certain class of operators in a slab domain is introduced and it is shown that that operator class is an algebra.

Key words: Dirichlet problems; Domains with a corner; Parametrices; Pseudo-differential operators.

1. Introduction. Let Ω be a domain with a corner in $\boldsymbol{R}^2$ given by

$$\Omega = \{(t\cos\theta, t\sin\theta)\ ;\ t \in \boldsymbol{R}_+,\ \varphi_1(t) < \theta < \varphi_2(t)\},$$

where $\varphi_1(t)$, $\varphi_2(t)$ are C^∞ functions on $\bar{\boldsymbol{R}}_+$ and $0 \le \varphi_1(t) < \varphi_2(t) < 2\pi$, $t \in \bar{\boldsymbol{R}}_+$, and $\varphi_1(0) = 0$, $\varphi_2(0) = \alpha$ and any derivative of $\varphi_j(e^x)$, $j = 1,2$, is bounded on $\boldsymbol{R}$. Let

$$\Gamma_j = \{(t\cos\varphi_j(t), t\sin\varphi_j(t))\ ;\ t \in \boldsymbol{R}_+\},\ j = 1,2.$$

We consider the following Dirichlet problem:

$$\text{(D)}\qquad \begin{cases} \Delta u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_1, \\ u = 0 \text{ on } \Gamma_2, \end{cases}$$

where f is a given function.

We consider (D) in polar coordinates and we map Ω into $\boldsymbol{R}_+ \times (0, \alpha)$ by an appropriate coordinate transformation. Moreover, by the change of variable $t = e^x$, we map $\boldsymbol{R}_+ \times (0, \alpha)$ into $\boldsymbol{R} \times (0, \alpha)$. Then we have the following Dirichlet problem $(\tilde{\mathrm{D}})$:

$$(\tilde{\mathrm{D}})\qquad \begin{cases} Lw = g \text{ in } \boldsymbol{R} \times (0, \alpha), \\ w = 0 \text{ on } \boldsymbol{R} \times \{0\}, \\ w = 0 \text{ on } \boldsymbol{R} \times \{\alpha\}, \end{cases}$$

where L is a strongly elliptic operator of second order in $\boldsymbol{R} \times [0, \alpha]$ and coefficients of L are real valued C^∞ functions in $\boldsymbol{R} \times [0, \alpha]$ whose derivatives of any order are bounded in $\boldsymbol{R} \times [0, \alpha]$ and the principal part of L is written in the form

$$L_0 = \partial_x^{\,2} + 2a_1(x, \theta)\partial_x\partial_\theta + a_2(x, \theta)\partial_\theta^{\,2}.$$

Since L is a strongly elliptic operator in $\boldsymbol{R} \times [0, \alpha]$ and coefficients of L_0 are real valued functions in $\boldsymbol{R} \times [0, \alpha]$, there exists a constant $\delta > 0$ such that

$$a_2(x, \theta) - a_1(x, \theta)^2 \ge \delta\ ;\quad x \in \boldsymbol{R},\ \theta \in [0, \alpha]. \tag{1.1}$$

The purpose of this paper is to construct a global parametrix for the problem $(\tilde{\mathrm{D}})$. For constructing a parametrix, we shall introduce a class of operators expressed by a sum of two integrals in two parameters in $[0, \alpha]$ of pseudo-differential operators on $\boldsymbol{R}$. This class of operators is closed under taking products of operators and taking formal adjoints. Those properties play a crucial role in the construction of a parametrix. Since we use only the condition (1.1) for constructing a parametrix, our method is applicable to general operators which have strong ellipticity instead of the Laplacian. Our parametrix is concerned with the Mellin transform and solutions of Dirichlet problems for the Laplacian in wedges in $\boldsymbol{R}^2$, cf. [1] and[7].

All the lemmas and theorems are stated without proofs.

2. A class of operators and its algebra. We shall use the following notations:

$\mathscr{S}(\boldsymbol{R})$ denotes the set of all rapidly decreasing functions on $\boldsymbol{R}$. For $w \in \mathscr{S}(\boldsymbol{R})\, \hat{w}$ denotes the Fourier transform of w. For $\xi \in \boldsymbol{R}$ we write $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. We say that $a(x, \xi) \in S^m_{1,0} (m \in \boldsymbol{R})$ when $a(x, \xi) \in C^\infty(\boldsymbol{R}^2)$ and for any γ_1, $\gamma_2 \ge 0$ there exists a constant $C_{\gamma_1,\gamma_2} > 0$ such that

$$|\partial_x^{\gamma_1}\partial_\xi^{\gamma_2} a(x, \xi)| \le C_{\gamma_1,\gamma_2}\langle\xi\rangle^{m-\gamma_2};\quad x, \xi \in \boldsymbol{R}.$$

We set $S^\infty_{1,0} = \cup_m S^m_{1,0}$. For $a(x, \xi) \in S^\infty_{1,0}$ we define a pseudo-differential operator $a(X, D_x)$

with symbol $a(x, \xi)$ by

$$(a(X, D_x)w(\cdot))(x) = \int_{-\infty}^{\infty} e^{ix\xi} a(x, \xi) \times \hat{w}(\xi)(2\pi)^{-1} d\xi, \ w \in \mathcal{S}(\boldsymbol{R}).$$

Op $S_{1,0}^{\infty}$ denotes the set of all pseudo-differential operators with symbols in $S_{1,0}^{\infty}$. For $a(x, \xi, \tau) \in S_{1,0}^{\infty}$ with a parameter τ, $a(X, D_x, \tau)$ denotes a pseudo-differential operator with symbol $a(x, \xi, \tau) \in S_{1,0}^{\infty}$ with a parameter τ.

First we introduce symbol classes $\mathcal{A}_{\alpha,j}^{m,\lambda}$; $m \in \boldsymbol{R}$, $\lambda \geq 0$, $j = 1,2$.

For $\alpha > 0$ we set $K_1 = \{(\theta, \mu); 0 \leq \theta \leq \alpha, 0 \leq \mu \leq \theta\}$, $K_2 = \{(\theta, \mu); 0 \leq \theta \leq \alpha, \theta \leq \mu \leq \alpha\}$.

Definition 2.1. (i) For $\lambda > 0$, $\mathcal{A}_{\alpha,j}^{m,\lambda}$ is the set of all $a(x, \xi, \theta, \mu) \in C^{\infty}(\boldsymbol{R}^2 \times K_j)$ such that for any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0$ and any $0 < \nu < \lambda$ there exists a constant $C_{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\nu} > 0$ such that

$$|\partial_x^{\gamma_1}\partial_\xi^{\gamma_2}\partial_\theta^{\gamma_3}\partial_\mu^{\gamma_4} a(x, \xi, \theta, \mu)| \leq C_{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\nu} \times \langle\xi\rangle^{m-\gamma_2+\gamma_3+\gamma_4} e^{-\nu|\mu-\theta|\langle\xi\rangle}; x, \xi \in \boldsymbol{R}, (\theta, \mu) \in K_j.$$

(ii) For $\lambda = 0$, $\mathcal{A}_{\alpha,j}^{m,0}$ is the set of all $a(x, \xi, \theta, \mu) \in C^{\infty}(\boldsymbol{R}^2 \times K_j)$ such that for any $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0$ there exists a constant $C_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} > 0$ such that

$$|\partial_x^{\gamma_1}\partial_\xi^{\gamma_2}\partial_\theta^{\gamma_3}\partial_\mu^{\gamma_4} a(x, \xi, \theta, \mu)| \leq C_{\gamma_1,\gamma_2,\gamma_3,\gamma_4,} \times \langle\xi\rangle^{m-\gamma_2+\gamma_3+\gamma_4}; x, \xi \in \boldsymbol{R}, (\theta, \mu) \in K_j.$$

A symbol $a(x, \xi, \theta, \mu) \in \mathcal{A}_{\alpha,j}^{m,\lambda}$ may be regarded as a symbol in $S_{1,0}^{m}$ with parameters θ, μ.

Next we introduce operator classes corresponding the above symbol classes.

Definition 2.2. A linear operator $A: C^0([0, \alpha]; \mathcal{S}(\boldsymbol{R})) \to C^0([0, \alpha]; \mathcal{S}(\boldsymbol{R}))$ is said to belong to the class Op $\mathcal{A}_{\alpha}^{m,\lambda}$, if there exist $a_j(x, \xi, \theta, \mu) \in \mathcal{A}_{\alpha,j}^{m,\lambda}$, $j = 1,2$, such that

$$(2.1) \quad Aw(\cdot, \theta) = \int_0^{\theta} a_1(X, D_x, \theta, \mu) w(\cdot, \mu) d\mu + \int_{\theta}^{\alpha} a_2(X, D_x, \theta, \mu) w(\cdot, \mu) d\mu.$$

For an operator $A \in$ Op $\mathcal{A}_{\alpha}^{m,\lambda}$ given in the form (2.1), we shall write $A = \mathrm{OP}(a_1, a_2)$.

We set Op $\mathcal{A}_{\alpha}^{\infty,\lambda} = \cup_m$ Op $\mathcal{A}_{\alpha}^{m,\lambda}$, Op $\mathcal{A}_{\alpha}^{-\infty,\lambda} = \cap_m$ Op $\mathcal{A}_{\alpha}^{m,\lambda}$.

For $A \in$ Op $\mathcal{A}_{\alpha}^{\infty,\lambda}$ its formal adjoint operator A^* should be defined by

$$\int_0^{\alpha} (Aw(\cdot, \theta), v(\cdot, \theta))_{L^2(\boldsymbol{R})} d\theta = \int_0^{\alpha} (w(\cdot, \theta), A^* v(\cdot, \theta))_{L^2(\boldsymbol{R})} d\theta;$$
$$w(\cdot, \theta), v(\cdot, \theta) \in C^0([0, \alpha]; \mathcal{S}(\boldsymbol{R})).$$

Now we state fundamental properties of Op $\mathcal{A}_{\alpha}^{\infty,\lambda}$.

Theorem 2.3. *The class* Op $\mathcal{A}_{\alpha}^{\infty,\lambda}$ *is an algebra in the following sense:*

(i) *If* $\lambda > 0$ *and* $A_j \in$ Op $\mathcal{A}_{\alpha}^{m_j,\lambda}$, $j = 1,2$, *then there exist* $B_k \in$ Op $\mathcal{A}_{\alpha}^{m_1+m_2-k-1,\lambda}$, $k \geq 0$, *such that* $A_1A_2 - \sum_{k=0}^{l-1} B_k \in$ Op $\mathcal{A}_{\alpha}^{m_1+m_2-l,0}$, $l \in \boldsymbol{N}$.

(ii) *If* $A_j \in$ Op $\mathcal{A}_{\alpha}^{m_j,0}$, $j = 1,2$, *then* $A_1A_2 \in$ Op $\mathcal{A}_{\alpha}^{m_1+m_2,0}$.

(iii) *If* $A \in$ Op $\mathcal{A}_{\alpha}^{m,\lambda}$, *then there exist* $C_k \in$ Op $\mathcal{A}_{\alpha}^{m-k,\lambda}$, $k \geq 0$, *such that* $A^* - \sum_{k=0}^{l-1} C_k \in$ Op $\mathcal{A}_{\alpha}^{m-l,0}$, $l \in \boldsymbol{N}$.

Lemma 2.4. *Let* $\lambda > 0$, $\mathrm{OP}(a_1, a_2) \in$ Op $\mathcal{A}_{\alpha}^{m,\lambda}$ *and* $c(x, \theta)$ *be a* C^{∞} *function in* $\boldsymbol{R} \times [0, \alpha]$ *whose derivative of any order is bounded in* $\boldsymbol{R} \times [0, \alpha]$. *Then for any* $w(\cdot, \theta) \in C^0([0, \alpha]; \mathcal{S}(\boldsymbol{R}))$ *we have*

$$(\mathrm{OP}(a_1, a_2)c)w(\cdot, \theta) = (\mathrm{OP}(a_1c(x, \theta), a_2c(x, \theta)) + A_1 + A_2)w(\cdot, \theta),$$

where $A_1 \in$ Op $\mathcal{A}_{\alpha}^{m-1,\lambda}$ *and* $A_2 \in$ Op $\mathcal{A}_{\alpha}^{m-2,0}$.

Theorem 2.3 and Lemma 2.4 suggest that calculus of Op$\mathcal{A}_{\alpha}^{\infty,\lambda}$ can be carried out by a similar way to the one of pseudo-differential operators.

In the proof of Theorem 2.3 we use Dirichlet's formula (cf. [5], p. 244).

3. Construction of a parametrix. First we construct a principal part of a global parametrix for the problem ($\tilde{\mathrm{D}}$).

We set

$$\lambda_1(x, \theta) = \frac{-a_1(x, \theta)}{a_2(x, \theta)},$$

$$\lambda_2(x, \theta) = \frac{\sqrt{a_2(x, \theta) - a_1(x, \theta)^2}}{a_2(x, \theta)}.$$

Since (1.1) and $a_j(x, \theta)$, $j = 1,2$, are real valued C^{∞} functions in $\boldsymbol{R} \times [0, \alpha]$ whose derivatives of any order are bounded in $\boldsymbol{R} \times [0, \alpha]$, $\inf\{\lambda_2(x, \theta); x \in \boldsymbol{R}, \theta \in [0, \alpha]\} > 0$ and $\lambda_1(x, \theta)$, $\lambda_2(x, \theta)$ and $\dfrac{1}{\lambda_2(x, \theta)}$ are real valued C^{∞} functions in $\boldsymbol{R} \times [0, \alpha]$ whose derivatives of any order are bounded in $\boldsymbol{R} \times [0, \alpha]$.

We set

$$\sigma = \inf\{\lambda_2(x, \theta); x \in \boldsymbol{R}, \theta \in [0, \alpha]\},$$

$$a_{1\pm}(x, \xi, \theta, \mu) = e^{\mu\lambda_2(x,\theta)\xi} \frac{e^{\pm(\theta-\alpha)\lambda_2(x,\theta)\xi}}{\sinh(\alpha\lambda_2(x, \theta)\xi)} \phi(\xi),$$

$$a_{2\pm}(x, \xi, \theta, \mu)$$

$$= e^{-\mu\lambda_2(x,\theta)\xi}\frac{e^{\pm(\theta-\alpha)\lambda_2(x,\theta)\xi}}{\sinh(\alpha\lambda_2(x,\theta)\xi)}\phi(\xi),$$

$$a_{3\pm}(x,\xi,\theta,\mu) = e^{(\mu-\alpha)\lambda_2(x,\theta)\xi}\frac{e^{\pm\theta\lambda_2(x,\theta)\xi}}{\sinh(\alpha\lambda_2(x,\theta)\xi)}\phi(\xi),$$

$$a_{4\pm}(x,\xi,\theta,\mu) = e^{-(\mu-\alpha)\lambda_2(x,\theta)\xi}\frac{e^{\pm\theta\lambda_2(x,\theta)\xi}}{\sinh(\alpha\lambda_2(x,\theta)\xi)}\phi(\xi),$$

where $\phi(\xi)$ is a cut off function at the origin.

Lemma 3.1. *We have*

$$a_{1\pm}(x,\xi,\theta,\mu),\ a_{2\pm}(x,\xi,\theta,\mu) \in \mathscr{A}^{0,\sigma}_{\alpha,1},$$

$$a_{3\pm}(x,\xi,\theta,\mu),\ a_{4\pm}(x,\xi,\theta,\mu) \in \mathscr{A}^{0,\sigma}_{\alpha,2}.$$

We set

$$q_1(x,\xi,\theta,\mu) = e^{i(\theta-\mu)\lambda_1(x,\theta)\xi} \times \sinh(\mu\lambda_2(x,\theta)\xi)\frac{\sinh((\theta-\alpha)\lambda_2(x,\theta)\xi)}{\sinh(\alpha\lambda_2(x,\theta)\xi)} \times \frac{1}{a_2(x,\theta)\lambda_2(x,\theta)\xi}\phi(\xi),$$

$$q_2(x,\xi,\theta,\mu) = e^{i(\theta-\mu)\lambda_1(x,\theta)\xi} \times \sinh((\mu-\alpha)\lambda_2(x,\theta)\xi)\frac{\sinh(\theta\lambda_2(x,\theta)\xi)}{\sinh(\alpha\lambda_2(x,\theta)\xi)} \times \frac{1}{a_2(x,\theta)\lambda_2(x,\theta)\xi}\phi(\xi),$$

where $\phi(\xi)$ is a cut off function at the origin. Then $q_j(x,\xi,\theta,\mu) \in \mathscr{A}^{-1,\sigma}_{\alpha,j}$, $j = 1,2$, are shown by using Lemma 3.1.

We define an operator $Q_0 \in \mathrm{Op}\,\mathscr{A}^{-1,\sigma}_{\alpha}$ by $Q_0 = \mathrm{OP}(q_1, q_2)$. Then we get the following:

Lemma 3.2. (i) *For any $w(\cdot,\theta) \in C^0([0,\alpha];\mathscr{S}(\boldsymbol{R}))$ we have $Q_0 w(\cdot,0) = 0$, $Q_0 w(\cdot,\alpha) = 0$, $Q_0 w(\cdot,\theta) \in C^2([0,\alpha];\mathscr{S}(\boldsymbol{R}))$ and*

$$(LQ_0)w(\cdot,\theta) = (\phi(D_x) - R_1)w(\cdot,\theta),$$

where $R_1 \in \mathrm{Op}\,\mathscr{A}^{0,\sigma}_{\alpha}$.

(ii) *For any $w(\cdot,\theta) \in C^2([0,\alpha];\mathscr{S}(\boldsymbol{R}))$ with $w(\cdot,0) = 0$ and $w(\cdot,\alpha) = 0$ we have*

$$(Q_0L)w(\cdot,\theta) = \Big(I + \frac{\phi(D_x)-1}{a_2(X,\theta)} \circ a_2(X',\theta) - R_2 - R_3\Big)w(\cdot,\theta),$$

where $R_2 \in \mathrm{Op}\,\mathscr{A}^{0,\sigma}_{\alpha}$ and $R_3 \in \mathrm{Op}\,\mathscr{A}^{-1,0}_{\alpha}$.

Lemma 3.2 suggests that the operator Q_0 plays a role of a principal part of a global parametrix for the problem ($\tilde{\mathrm{D}}$). In the proof of (ii) of Lemma 3.2 we use Lemma 2.4.

We shall now construct a parametrix on the base of Lemma 3.2. First note that an analogous theorem to Hörmander's one (cf. [2], Chapter 2, Lemma 3.2) holds over $\mathscr{A}^{\infty,0}_{\alpha,j}$, $j = 1,2$. Moreover, by using (i) and (ii) of Theorem 2.3 we have $R_1{}^k, (R_2 + R_3)^k \in \mathrm{Op}\,\mathscr{A}^{-k+1,0}_{\alpha}$, $k \in \boldsymbol{N}$.

From two results stated above, it follows that there exists $E_1 \in \mathrm{Op}\,\mathscr{A}^{0,0}_{\alpha}$ such that $E_1 \sim \sum_{k=0}^{\infty} R_1{}^k$ (modulo $\mathrm{Op}\,\mathscr{A}^{-\infty,0}_{\alpha}$) and there exists $E_2 \in \mathrm{Op}\,\mathscr{A}^{0,0}_{\alpha}$ such that $E_2 \sim \sum_{k=0}^{\infty}(R_2 + R_3)^k$ (modulo $\mathrm{Op}\,\mathscr{A}^{-\infty,0}_{\alpha}$). Therefore we obtain a parametrix as follows:

Theorem 3.3. (i) *There exist $E_1 \in \mathrm{Op}\,\mathscr{A}^{0,0}_{\alpha}$ and $K_1 \in \mathrm{Op}\,\mathscr{A}^{-\infty,0}_{\alpha}$ such that for any $w(\cdot,\theta) \in C^0([0,\alpha];\mathscr{S}(\boldsymbol{R}))$ we have*

$$(LQ_0E_1 - I)w(\cdot,\theta) = K_1 w(\cdot,\theta).$$

(ii) *There exist $E_2 \in \mathrm{Op}\,\mathscr{A}^{0,0}_{\alpha}$ and $k_2 \in \mathrm{Op}\,\mathscr{A}^{-\infty,0}_{\alpha}$ such that for any $w(\cdot,\theta) \in C^2([0,\alpha];\mathscr{S}(\boldsymbol{R}))$ with $w(\cdot,0) = 0$ and $w(\cdot,\alpha) = 0$ we have*

$$(E_2Q_0L - I)w(\cdot,\theta) = K_2 w(\cdot,\theta).$$

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