Totally Real Minimal Submanifolds in a Quaternion Projective Space

By LIU Ximin

Department of Mathematics, Nankai University, China

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Abstract: In this paper, we prove some pinching theorems with respect to the scalar curvatures of 4-dimensional projectively flat (conharmonically flat) totally real minimal submanifolds in a 16-dimensional quaternion projective space.

Keywords: Totally real submanifold, Quaternion projective space, Curvature pinching

1. Introduction. A quaternion Kaehler manifold is defined as a $4n$-dimensional Riemannian manifold whose holonomy group is a subgroup of $Sp(1)\cdot Sp(n)$. A quaternion projective space $QP^n(c)$ is a quaternion Kaehler manifold with constant quaternion sectional curvature $c>0$.

Let M be an n-dimensional Riemannian manifold and $J: M \to QP^n(c)$ an isometric immersion of M into $QP^n(c)$. If each tangent 2-subspace of M is mapped by J into a totally real plane of $QP^n(c)$, then M is called a totally real submanifold of $QP^n(c)$. Funabashi [3], Chen and Houh [1] and Shen [6] studied this submanifold and got some curvature pinching theorems. The purpose of this paper is to give some characterizations of 4-dimensional projectively flat (conharmonically flat) totally real minimal sub-manifolds in $QP^4(c)$.

2. Preliminaries. Let $QP^n(c)$ denote a $4n$-dimensional quaternion projective space with constant quaternion sectional curvature $c>0$ and M be a totally real minimal submanifold in $QP^n(c)$ of dimension n. In this paper we will use the same notations and terminologies as in [1]. It was proved in [1] that the second fundamental form of the immersion satisfies

$$(2.1)\quad \frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum tr(A_uA_v - A_vA_u)^2 - \sum (tr\,A_uA_v)^2 + \frac{c}{4}(n+1)\|\sigma\|^2$$

Since $\sum tr(A_uA_v - A_vA_u)^2$

$$= -\sum_{u,v,k,l}\Big(\sum_m (h^u_{km}h^v_{lm} - h^v_{km}h^u_{lm})\Big)^2$$

this together with the equation of Gauss, implies

$$(2.2)\quad \sum tr(A_uA_v - A_vA_u)^2 = -\|R\|^2 + c\rho - \frac{n-1}{8}nc^2.$$

Similarly, we have

$$(2.3)\quad \sum (tr\,A_uA_v)^2 = \|S\|^2 - \frac{n-1}{2}c\rho + n\Big(\frac{n-1}{4}c\Big)^2.$$

Combining (2.1) with (2.2), (2.3) and $\|\sigma\|^2 = \frac{c}{4}n(n-1) - \rho$, we obtain

$$(2.4)\quad \frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \|R\|^2 - \|S\|^2 + \frac{n+1}{4}c\rho.$$

3. Projectively flat totally real minimal submanifold. Suppose M is an n-dimensional compact oriented totally real minimal submanifold in $QP^n(c)$, if M is projectively flat, then its projective curvature tensor $P^{[2]}$ satisfies

(3.1) $P(X,Y,Z,W) \overset{\mathrm{def}}{=} R(X,Y,Z,W) - (g(X,W)S(Y,Z) - g(Y,W)S(X,Z))/(n-1) = 0$, where R, S, g are the curvature tensor, Ricci tensor and Riemannian metric of M respectively. From (3.1) we have

$$(3.2)\quad \|R\|^2 = \frac{2}{n-1}\|S\|^2$$

which, together with (2.4) asserts

$$(3.3)\quad \frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 + \frac{n+1}{n-1}\|S\|^2 + \frac{n+1}{4}c\rho.$$

Taking the integrals of the both sides of (3.3) and using Green's theorem, we have

$$(3.4)\quad \int_M \|\nabla'\sigma\|^2 dV = \int_M \Big(\|S\|^2/(n-1) - \frac{c}{4}\rho\Big)(n+1)dV$$

On the other hand, by the Gauss-Bonnet theorem, when $n=4$, the Euler number $\chi(M)$ of

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M is given by

(3.5) $$\chi(M) = \frac{1}{32\pi^2}\int_M (\|R\|^2 - 4\|S\|^2 + \rho^2)dV.$$

From (3.2), (3.4) and (3.5) we get

(3.6) $$2\int_M \|\nabla'\sigma\|^2 dV + 32\pi^2\chi(M) = \int_M \rho\left(\rho - \frac{5}{2}c\right)dV$$

when $\chi(M)$ is nonnegative, from $\rho > 0$ we can derive $\rho \geq \frac{5}{2}c$, then from Theorem 4 of [1] or Remark 3.1 of [5], we can obtain the following theorem.

Theorem A. Let M be a 4-dimensioanl compact oriented projectively flat totally real minimal submanifold in $QP^4(c)$. If M has nonnegative Euler number and the scalar curvature $\rho > 0$, then M is totally geodesic.

4. Conharmonically flat totally real minimal submanifold. Suppose that M is an n-dimensional compact oriented totally real minimal submanifold in $QP^n(c)$. If M is conharmonically flat, then its conharmonic curvature tensor C[4] satisfies

(4.1) $$C(X, Y, Z, W) \overset{\text{def}}{=} R(X, Y, Z, W) - (g(X, W)S(Y, Z) - g(Y, W)S(X, Z) + g(Y, Z)S(X, W) - g(X, Z)S(Y, W))/(n-2).$$

From (4.1) we can get

(4.2) $$\|R\|^2 = (4\|S\|^2 + 2\rho^2)/(n-2)$$

which, together with (2.4) asserts

(4.3) $$-\frac{1}{2}\|\Delta\sigma\|^2 = \frac{n+2}{n-2}\|S\|^2 + \frac{2}{n-2}\rho^2 - (n+1)\frac{c}{4}\rho - \|\nabla'\sigma\|^2.$$

Taking the integrals of the both sides of (4.3) and using Green's theorem, we have

(4.4) $$\int_M \|\nabla'\sigma\|^2 dV = \int_M \left(\frac{n+2}{n-2}\|S\|^2 + \frac{2}{n-2}\rho^2 - (n+1)\frac{c}{4}\rho\right)dV$$

when $n = 4$, from (3.5), (4.3) and (4.4) we have

(4.5) $$48\pi^2\chi(M) + \int_M \|\nabla'\sigma\|^2 dV = \int_M \rho\left(4\rho - \frac{5}{4}c\right)dV.$$

So we can get the following Theorem immediately.

Theorem B. Let M be a 4-dimensional compact oriented conharmonically flat totally real minimal submanifold in $QP^4(c)$. If M has nonnegative Euler number and the scalar curvature ρ of M satisfies $0 \leq \rho \leq \frac{5c}{16}$, then either $\rho = 0$ or $\rho = \frac{5c}{16}$.

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