On Bloch-to-Besov Composition Operators

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1. Introduction. Let $H(D)$ be the space of analytic functions on the unit disk D. Every holomorphic self-map $\varphi : D \to D$ induces a linear composition operator C_φ from $H(D)$ into itself as follows: $C_\varphi f = f \circ \varphi$, whenever $f \in H(D)$. In this paper we consider composition operators from the Bloch space $\mathscr{B}$ to the spaces of analytic Besov functions B_p, $1 < p < \infty$.

Recall the definitions of the Bloch space $\mathscr{B}$ and the analytic Besov spaces B_p (see e.g. [9]).

The function f is called a Bloch function if it is analytic in D and

$$\| f \|_{\mathscr{B}} = \sup_{z \in D} (1 - |z|^2) \, | f'(z) | < \infty.$$

This defines a semi-norm. The Bloch functions form a Banach space $\mathscr{B}$ with the norm $\| f \| = | f(0) | + \| f \|_{\mathscr{B}}$.

The analytic Besov functions are defined as follows

$$B_p = \left\{ f \in H(D) : \| f \|_{B_p} = \left(\iint_D ((1 - |z|^2) \, | f'(z) |)^p d\lambda(z) \right)^{\frac{1}{p}} < \infty \right\},$$

where $d\lambda(z) = \dfrac{dA(z)}{(1 - |z|^2)^2}$ is the hyperbolic area measure on D and $dA(z) = \dfrac{1}{\pi} dx\, dy$.

The analytic Besov functions form a Banach space B_p, $1 < p < \infty$, with the norm $\| f \| = | f(0) | + \| f \|_{B_p}$.

Let B be the family of holomorphic self-maps φ of the unit disk D into itself. By the Schwarz-Pick lemma $\sup_{z \in D} (1 - |z|^2) \varphi^*(z) \leq 1$ for any $\varphi \in B$, where $\varphi^*(z)$ is the hyperbolic derivative

$$\varphi^*(z) = \frac{| \varphi'(z) |}{1 - | \varphi(z) |^2}.$$

We say that $\varphi \in B_0$ if

$$\lim_{|z| \to 1} (1 - |z|^2) \varphi^*(z) = 0.$$

Definition. *For $1 < p < \infty$ the hyperbolic analytic Besov class B_p^h is defined to be the family of all functions $\varphi \in B$ such that*

$$\| \varphi \|_{B_p^h} = \left(\iint_D ((1 - |z|^2) \varphi^*(z))^p d\lambda(z) \right)^{\frac{1}{p}} < \infty.$$

We can assume that $B_\infty^h = B$. However Möbius transforms of D are not p-hyperbolic Besov functions for $1 < p < \infty$.

Let $T_a(z) = \dfrac{a - z}{1 - \bar{a}z}$, $a \in D$, and $\varphi_a(z) = \varphi(T_a(z))$.

For every $\varphi \in B_p^h$ and every $a \in D$ functions $T_a \circ \varphi(z)$ and $\varphi \circ T_a(z)$ belong to the class B_p^h.

We give some examples of functions which are in B_p^h or are not in B_p^h.

1. Let $S_\alpha = \{ z = x + iy : |x|^\alpha + |y|^\alpha < 1 \}$, $0 < \alpha \leq 1$, and φ_α be a conformal mapping of D into S_α, then $\varphi_1 \notin B_2^h$. If $\alpha < 1$ then $\varphi_\alpha(z) \in B_2^h$.

2. Let φ be a bounded holomorphic function in D with $\| \varphi \|_\infty \leq k < 1$. If φ is continuous in $\bar{D}$ and $\varphi(e^{i\theta}) \in \Lambda_\alpha$, $0 < \alpha \leq 1$, then
$(1 - |z|^2) \, | \varphi'(z) | = O((1 - |z|^2)^\alpha)$ as $|z| \to 1$, and also
$(1 - |z|^2) \varphi^*(z) = O((1 - |z|^2)^\alpha)$ as $|z| \to 1$
by the Hardy-Littlewood theorem ([3], Theorem 5.1). Thus $\varphi \in B_p^h$ for $p > \dfrac{1}{\alpha}$. See also [7].

2. Composition operators on the Bloch space. Our main results are the following

Theorem 1. *Let φ be a holomorphic mapping of D into itself and $1 < p < \infty$. C_φ is a Bloch-to-B_p composition operator if and only if $\varphi \in B_p^h$.*

Theorem 2. *If $\varphi \in B_p^h$, $1 < p < \infty$, then φ induces a compact composition operator C_φ on $\mathscr{B}$ into B_p.*

Proof of Theorem 1. Let φ be any function of B_p^h, $1 < p < \infty$, and f be any Bloch function. Then we obtain

$$\| f \circ \varphi \|_{B_p}^p = \iint_D (1 - |z|^2)^p \, | g'(z) |^p d\lambda(z)$$

$$= \iint_D (1 - |z|^2)^p \, | f' \circ \varphi(z) |^p \, | \varphi'(z) |^p d\lambda(z)$$

$$= \iint_D (1-|z|^2)^p (\varphi^*(z))^p (1-|\varphi(z)|^2)^p$$
$$|f' \circ \varphi(z)|^p d\lambda(z) \leq \|f\|_{\mathscr{B}}^p \|\varphi\|_{B_p^h}^p < \infty.$$

This proves the "if" part.

To prove the converse we use a trick in [2]. Pick up any Bloch functions f and g such that $|f'(z)| + |g'(z)| \geq \dfrac{1}{1-|z|^2}$ (existence of such functions was proved in [5]). Then for every $p > 1$

$$|f'(z)|^p + |g'(z)|^p \geq \frac{2^{1-p}}{(1-|z|^2)^p}$$

and hence $2^{1-p}\|\varphi\|_{B_p^h}^p \leq \|f \circ \varphi\|_{B_p}^p + \|g \circ \varphi\|_{B_p}^p < \infty$. □

Proof of Theorem 2. Let $b(\mathscr{B})$ be the unit ball in $\mathscr{B}$ and $\{f_n\} \subset b(\mathscr{B})$. Then $\{f_n\}$ is a normal family in D and therefore there is a subsequence $\{f_{n_k}\}$ converging uniformly on every compact subset of D to $f \in b(\mathscr{B})$. Then the sequence $\{g_k\}$, $g_k(z) = f_{n_k}(z) - f(z)$, converges uniformly to 0 on every compact subset of D. Thus for compactness of operator $C_\varphi : \mathscr{B} \to B_p$ it is enough to prove that if $\{g_k\} \in b(\mathscr{B})$ and $\{g_k\}$ converges to 0 uniformly on every compact subset of D then $\lim_{k\to\infty} \|g_k \circ \varphi\|_{B_p} = 0$.

Let $\{g_k\} \in b(\mathscr{B})$ and $\{g_k\}$ converge to 0 uniformly on every compact subset of D. Since $\varphi \in B_p^h$ for every $\varepsilon > 0$ there exists such a compact $K \subset D$ that

$$\iint_{D\setminus K} (1-|z|^2)^p (\varphi^*(z))^p d\lambda(z) < \varepsilon$$

and there exists a number N such that $\sup_{w\in\varphi(K)} (1-|w|^2)\,|g_k'(w)| < \varepsilon^{\frac{1}{p}}$ for any $k \geq N$. Then

$$\|g_k \circ \varphi\|_{B_p}^p = \iint_D (1-|z|^2)^p |(g_k \circ \varphi)'(z)|^p d\lambda(z)$$
$$= \iint_K (1-|z|^2)^p (\varphi^*(z))^p (1-|\varphi(z)|^2)^p |g_k' \circ \varphi(z)|^p d\lambda(z)$$
$$+ \iint_{D\setminus K} (1-|z|^2)^p (\varphi^*(z))^p (1-|\varphi(z)|^2)^p |g_k' \circ \varphi(z)|^p d\lambda(z)$$
$$\leq \varepsilon \iint_K (1-|z|^2)^p (\varphi^*(z))^p d\lambda(z) + 1\cdot \iint_{D\setminus K} (1-|z|^2)^p (\varphi^*(z))^p d\lambda(z)$$
$$\leq \varepsilon \|\varphi\|_{B_p^h}^p + \varepsilon = \varepsilon \cdot const.$$ □

Corollary. *Let φ be any holomorphic self-map of the unit disk D. Then $\varphi \in B_p^h$, $1 < p < \infty$, if and only if*

$$(1) \qquad \iint_D \iint_D \frac{|f\circ\varphi(z) - f\circ\varphi(w)|^p}{|1 - z\bar{w}|^4} dA(z)\,dA(w) < \infty$$

for any Bloch function f.

Proof. Let $1 < p < \infty$. If $\varphi \in B_p^h$ then by Theorem 1 for any $f \in \mathscr{B}$ function $f\circ\varphi \in B_p$ and by Theorem 5.3.4 [9] we have (1).

Conversely, if (1) holds for every Bloch function f, then by Theorem 5.3.4 [9] function $f\circ\varphi$ belongs to B_p and by Theorem 1 function $\varphi \in B_p^h$.

3. Properties of hyperbolic Besov functions. In this section we give some properties of hyperbolic Besov functions.

Denote by $\sigma(a, b)$ the hyperbolic distance on D

$$\sigma(a,b) = \frac{1}{2}\ln\frac{|1-a\bar{b}| + |a-b|}{|1-a\bar{b}| - |a-b|}, \quad a, b \in D.$$

Theorem 3. *Let φ be any holomorphic self-map of the unit disk D. Then $\varphi \in B_p^h$, $1 < p < \infty$, if and only if*

$$\iint_D \iint_D \frac{\sigma(\varphi(z), \varphi(w))^p}{|1 - z\bar{w}|^4} dA(z)\,dA(w) < \infty.$$

Theorem 4. $B_p^h \subset B_q^h \subset B_0$ *for any* $1 < p < q < \infty$.

Remark 1. The similar result to Theorem 3 for analytic Besov functions was proved by K. Zhu ([9], Theorem 5.3.4).

Remark 2. Recently, R. Aulaskari and G. Csordas defined the meromorphic (spherical) Besov classes $B_p^{\#}$, $1 < p < \infty$, (see [1]). Similar to Theorem 1 we can show that if f is a normal meromorphic function in D [4] and $\varphi \in B_p^h$ then $f\circ\varphi \in B_p^{\#}$.

Addendum. Prof. T. Gamelin informed the author that Maria Tjani [6] independently obtained similar results to Theorem 1 and Theorem 2.

Prof. R. Aulaskari informed the author that his student Ruhan Zhao [8] also obtained similar results to Theorem 1 and Theorem 2. All these proofs are different.

References

[1] R. Aulaskari and G. Csordas: Besov spaces and the $Q_{q,0}$ classes. Acta Sci. Math. (Szeged), **60**, nos. 1–2, 31–48 (1995).

[2] B. R. Choe, W. Ramey, and D. Ullrich: Bloch-to-BMOA pullbacks on the disk (preprint).

[3] R. Duren: Theory of H^p spaces. Academic Press, New York (1970).

[4] K. Noshiro: Contributions to the theory of meromorphic functions in the unit circle. J. Fac. Sci. Hokkaido Univ., **7**, 149–159 (1938).

[5] W. Ramey and D. Ullrich: Bounded mean oscillations of Bloch pullbacks. Math. Ann., **291**, 591–606 (1991).

[6] M. Tjani: Compact composition operators on some Möbius invariant Banach spaces. Thesis, Michigan State University (1996).

[7] S. Yamashita: Smoothness of the boundary values of functions bounded and holomorphic in the disk. Trans. Amer. Math. Soc., **272**, 539–544 (1982).

[8] R. Zhao: Composition operators from Bloch type spaces to Hardy and Besov spaces (preprint).

[9] K. Zhu: Operator theory in function spaces. Pure and Applied Mathematics, Marcel Dekker, New York (1990).