Convergence in the Space of Fourier Hyperfunctions

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Abstract: A structural characterization of a convergent family of Fourier hyperfunctions $\{f_h ; h \in \Gamma\}$ is given.

1. Notations and definitions. We denote by D^n the compactification of R^n, $D^n = R^n \cup S_\infty^{n-1}$ and supply it with the usual topology. The sheaves $\tilde{\mathcal{O}}$ and $\mathcal{Q}$ on $D^n + iR^n$ are defined as follows (cf. [3-6]). For any open set $U \subset D^n + iR^n$, $\tilde{\mathcal{O}}(U)$ consists of those elements of $\mathcal{O}(U \cap C^n)$ which satisfy $|F(z)| \leq C_{V,\varepsilon} \exp(\varepsilon |\operatorname{Re} z|)$ uniformly for any open set $V \subset C^n$, $\bar{V} \subset U$, and for every $\varepsilon > 0$. Hence, $\tilde{\mathcal{O}}|_{C^n} = \mathcal{O}$. The derived sheaf $\mathcal{H}^n_{D^n}(\tilde{\mathcal{O}})$, denoted by $\mathcal{Q}$, is called the sheaf of Fourier hyperfunctions. It is a flabby sheaf on D^n ([4]).

Let I be a convex neighbourhood of $0 \in R^n$ and $U_j = \{(D^n + iI) \cap \{\operatorname{Im} z_j \neq 0\}\}, j = 1, \ldots, n$. The family $\{D^n + iI, U_j ; j = 1, \ldots, n\}$ gives a relative Leray covering for the pair $\{D^n + iI, (D^n + iI) \setminus D^n\}$ relative to the sheaf $\tilde{\mathcal{O}}$. Thus $\mathcal{Q}(D^n) = \tilde{\mathcal{O}}((D^n + iI) \# D^n) / \sum_{j=1}^n \tilde{\mathcal{O}}((D^n + iI) \#_j D^n)$, where $(D^n + iI) \# D^n = U_1 \cap \ldots \cap U_n$ and $(D^n + iI) \#_j D^n = U_1 \cap \ldots \cap U_{j-1} \cap U_{j+1} \cap \ldots \cap U_n$.

We shall use the notation Λ for the set of n-vectors with entry $\{-1,1\}$; the corresponding open orthants in R^n will be denoted by Γ_σ, $\sigma \in \Lambda$.

A global section $f = [F] \in \mathcal{Q}(D^n)$ is defined by $F \in \tilde{\mathcal{O}}((D^n + iI) \# D^n)$; $F = (F_\sigma)$, where $F_\sigma \in \tilde{\mathcal{O}}(D^n + iI_\sigma)$, $D^n + iI_\sigma$ is an infinitesimal wedge of type $R^n + i\Gamma_\sigma 0$, $\sigma \in \Lambda$.

Recall the topological structure of $\mathcal{Q}(D^n)$. Let $f = [F] \in \mathcal{Q}(D^n)$, $F \in \tilde{\mathcal{O}}(D^n + iI) \# D^n)$. Then, by $P_{K,\varepsilon}(F) = \sup_{z \in R^n + iK} |F(z) \exp(-\varepsilon |\operatorname{Re} z|)|$, $\varepsilon > 0$, $K \Subset I \setminus \{0\}$, is defined the family of semi-norms; $\tilde{\mathcal{O}}((D^n + iI) \# D^n)$ is a Fréchet and Montel space, as well as $\mathcal{Q}(D^n)$.

Let $f = [F] \in \mathcal{Q}(D^n)$. Then we associate to f, $f(x) \cong \sum_{\sigma \in \Lambda} sgn\sigma\, F_\sigma(x + i\Gamma_\sigma 0)$, $F_\sigma \in \tilde{\mathcal{O}}(D^n + iI_\sigma)$ (cf. [3], Theorem 8.5.3 and Definition 8.3.1).

The Fourier transform on $\mathcal{Q}(D^n)$ is defined by the use of functions $\chi_\sigma = \chi_{\sigma_1} \ldots \chi_{\sigma_n}$, where $\sigma_k = \pm 1$, $k = 1, \ldots, n$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\chi_1(t) = e^t / (1 + e^t)$, $\chi_{-1}(t) = 1/(1 + e^t)$, $t \in R$. Let $u(x) \cong \sum_{\sigma \in \Lambda} U_\sigma(x + i\Gamma_\sigma 0) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} (\chi_{\tilde{\sigma}} U_\sigma)(x + i\Gamma_\sigma 0)$, where $\chi_{\tilde{\sigma}} U_\sigma \in \mathcal{O}(D^n + iI_\sigma)$, σ, $\tilde{\sigma} \in \Lambda$ and decreases exponentially along the real axis outside the closed $\tilde{\sigma}$-th orthant.

The Fourier transform of u is defined by

$$\mathcal{F}(u) \cong \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma)(x - i\Gamma_{\tilde{\sigma}} 0)$$

$$= \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \int_{\operatorname{Im} z = y^k} e^{-iz\zeta} (\chi_{\tilde{\sigma}} U_\sigma)(z) dx, \; y^k \in I_\sigma,$$

where $\mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma) \in \tilde{\mathcal{O}}(D^n - iI_{\tilde{\sigma}})$ and $\mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma)$ decreases exponentially along the real axis outside the closed σ-orthant.

An infinite-order differential operator $J(D) = \sum_{|\alpha| \geq 0} b_\alpha D^\alpha$ with $\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|b_\alpha| \alpha!} = 0$ is called a local operator.

2. Convergence in $\mathcal{Q}(D^n)$. Let E be a Fréchet space with an increasing family of seminorms $\{P_i ; i \in N\}$ and let F be a closed subspace of E. Denote by $\tilde{x}$ an element of the quotient space E/F defined by $x \in E$; seminorms which induce the topology in E/F are given by $p_i(\tilde{x}) = \inf_{y \in F} P_i(x + y)$, $i \in N$. In the sequel Γ will be a convex cone in R^n.

Proposition 1. *A necessary and sufficient condition that a family $\{\tilde{x}_h ; h \in \Gamma\}$ converges to $\tilde{x}$ in E/F as $\|h\| \to \infty$, $h \in \Gamma$, is the existence of a family $\{u_h \in E ; h \in \Gamma\}$ such that u_h belongs to the class $\tilde{x}_h$ for every $h \in \Gamma$ and u_h converges to u in E as $\|h\| \to \infty$, $h \in \Gamma$, where u belongs to the class $\tilde{x}$.*

Proof. The sufficiency is trivial. Suppose that $\tilde{x}_h$ converges to $\tilde{x}$ in E/F as $\|h\| \to \infty$, $h \in \Gamma$. Then for every $m \in N$ there exists $t_m > 0$ such that $p_m(\tilde{x}_h - \tilde{x}) = \inf_{y \in F} P_m(x_h - x + y) < 1/m$, $\|h\| \geq t_m$, $h \in \Gamma$; $\{t_m ; m \in N\}$ is a monotone increasing sequence which tends to infinity as $m \to \infty$. We construct a looked-for

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family $\{u_h ; h \in \Gamma\}$ as follows. For every $h \in \Gamma$, $\| h \| \geq t_{m_0}$, there exists $y_{m_0,h} \in F$ such that $P_{m_0}(x_h - x + y_{m_0,h}) < 2/m_0$, $\| h \| \geq t_{m_0}$, $h \in \Gamma$.

Then $u_h = x_h + y_{m,h}$, for those $h \in \Gamma$ for which $t_m \leq \| h \| < t_{m+1}$, $m \in N$. The verification of the assertion simply follows.

This proposition implies the next one.

Proposition 2. *Let I be a convex neighbourhood of $0 \in R^n$ and $I_\sigma = I \cap \Gamma_\sigma$, $\sigma \in \Lambda$. Let $\{f_h ; h \in \Gamma\}$ be a family in $\mathcal{Q}(D^n)$ such that $f_h \cong \sum_{\sigma\in\Lambda} G_{h,\sigma}(x + i\Gamma_\sigma 0)$, where $G_{h,\sigma} \in \tilde{\mathcal{O}}(D^n + iI_\sigma)$, $h \in \Gamma$, $\sigma \in \Lambda$.*

A necessary and sufficient condition that f_h converges in $\mathcal{Q}(D^n)$ to $f \cong \sum_{\sigma\in\Lambda} G_\sigma(x + i\Gamma_\sigma 0)$ as $\| h \| \to \infty$, $h \in \Gamma$ is the existence of families $\{F_{h,\sigma} ; h \in \Gamma\} \subset \tilde{\mathcal{O}}(D^n + iI_\sigma)$, $\sigma \in \Lambda$, such that

1) $F_h = (F_{h,\sigma})$ belongs to the same class as $G_h = (G_{h,\sigma})$, $h \in \Gamma$;

2) For every $\sigma \in \Lambda$, $F_{h,\sigma}$ converges to F_σ in $\tilde{\mathcal{O}}(D^n + iI_\sigma)$ as $\| h \| \to \infty$, $h \in \Gamma$,

where $F = (F_\sigma)$ belongs to the same class as $G = (G_\sigma)$.

One can find in [3, p. 408] the sufficiency of the given condition.

Theorem 1. *Let $\{f_h ; h \in \Gamma\}$ be a family in $\mathcal{Q}(D^n)$ of the form $f_h = [G_h]$, $G_h \in \tilde{\mathcal{O}}((D^n + iI) \# D^n)$, $h \in \Gamma$, such that f_h converges to f in $\mathcal{Q}(D^n)$ as $\| h \| \to \infty$, $h \in \Gamma$.*

Then for every sequence $\{h_\nu ; \nu \in N\}$ in Γ, such that $\| h_\nu \| \to \infty$ as $\nu \to \infty$, there exists an elliptic local operator $J(D)$ and a sequence of functions $\{q_{h_\nu} ; h_\nu \in \Gamma, \nu \in N\}$ with the properties:

a) For every $\varepsilon > 0$ there exists $C_{h_\nu,\varepsilon} > 0$, $\nu \in N$ such that $| q_{h_\nu}(x) | \leq C_{h_\nu,\varepsilon} \exp(\varepsilon | x |)$, $x \in R^n$, $\nu \in N$ (q_{h_ν} is slowly increasing). Thus q_{h_ν} defines an element of $\mathcal{Q}(D^n)$, denoted by ℓq_{h_ν} for every $h_\nu \in \Gamma$.

b) q_{h_ν} converges in $\mathcal{E}(R^n)$ to q as $\| h_\nu \| \to \infty$, $h_\nu \in \Gamma$, $\nu \to \infty$, where q is also slowly increasing and defines $\ell q \in \mathcal{Q}(D^n)$. Moreover, for every ε.

$$\sup_{x\in R^n} \| q_{h_\nu}(x) - q(x) \| e^{-\varepsilon|x|} \to 0, \ \nu \to \infty.$$

c) $f_{h_\nu} = J(D)\ell q_{h_\nu}$, $h_\nu \in \Gamma$, $\nu \in N$ and $f = J(D)\ell q$.

(Note, q in b) and c) is the same for every sequence $\{q_{h_\nu} ; h_\nu \in \Gamma, \nu \in N\}$).

Proof. We will use some ideas of Kaneko's papers [1] and [2]. Proposition 2 implies that there exists a family $\{\tilde{F}_h ; h \in \Gamma\} \subset \tilde{\mathcal{O}}((D^n + iI) \# D^n)$ such that $f_h = [\tilde{F}_h]$, $f = [\tilde{F}]$ and $\tilde{F}_h$ converges to $\tilde{F}$ in $\tilde{\mathcal{O}}((D^n + iI) \# D^n)$.

Let $f_n \cong \sum_{\sigma\in\Lambda} F_{h,\sigma}(x + i\Gamma_\sigma 0)$, where $F_{h,\sigma} = \mathrm{sgn}\sigma \tilde{F}_{h,\sigma}$. Its Fourier transform is defined by

$$\mathcal{F} f_h \cong \sum_{\sigma\in\Lambda} \sum_{\tilde{\sigma}\in\Lambda} \mathcal{F}(\chi_{\tilde{\sigma}} F_{h,\sigma})(\xi - i\Gamma_{\tilde{\sigma}} 0) = \sum_{\sigma\in\Lambda} \sum_{\tilde{\sigma}\in\Lambda} R_{h,\sigma,\tilde{\sigma}}(\xi - i\Gamma_{\tilde{\sigma}} 0),$$

where $R_{h,\sigma,\tilde{\sigma}} \in \tilde{\mathcal{O}}(D^n - iI_{\tilde{\sigma}})$ and $R_{h,\sigma,\tilde{\sigma}}$ decreases exponentially along the real axis outside the closed σ-th orthant for every $h \in \Gamma$. By Proposition 2, $R_{h,\sigma,\tilde{\sigma}}$ converges in $\tilde{\mathcal{O}}(D^n - iI_{\tilde{\sigma}})$ to $R_{\sigma,\tilde{\sigma}} = \mathcal{F}(\chi_{\tilde{\sigma}} F_\sigma)$ as $\| h \| \to \infty$, $h \in \Gamma$.

Let $\{h_\nu ; \nu \in N\}$ be a sequence in Γ such that $\| h_\nu \| \to \infty$ as $\nu \to \infty$. Since $\tilde{\mathcal{O}}(D^n - iI_{\tilde{\sigma}})$ is a Montel space, the set $A_{\sigma,\tilde{\sigma}} = \{R_{\sigma,\tilde{\sigma}}, R_{h_\nu,\sigma,\tilde{\sigma}} ; h_\nu \in \Gamma, \nu \in N\}$ is a compact set in $\tilde{\mathcal{O}}(D^n - iI_{\tilde{\sigma}})$ for every σ and $\tilde{\sigma}$.

First we shall prove the existence of a sequence $\{\varphi_j ; j \in N\}$ of positive monotone increasing functions defined on $R_+ \cup \{0\}$ and a sequence of positive constants $\{C_j ; j \in N\}$ which do not depend on σ, $\tilde{\sigma} \in \Lambda$ and $h_\nu \in \Gamma$, $\nu \in N$ and:

(1) a) $\varphi_j(0) = 1$, $\varphi_j(r) \to \infty$ as $r \to \infty$, $j \in N$;
b) $| R_{h_\nu,\sigma,\tilde{\sigma}}(\zeta) | \leq C_j \exp(| \zeta | / \varphi_j(| \zeta |)$, $\zeta \in R^n + iK_{\tilde{\sigma},j}$,

where $\{K_{\tilde{\sigma},j} ; j \in N\}$ is a sequence of compact sets which exhausts $- I_{\tilde{\sigma}}$ from the inside.

Let $j \in N$. Put

$$B_j(r) = \sup_{\sigma,\tilde{\sigma}} \ \sup_{|\zeta|=r, \zeta\in R^n+iK_{\tilde{\sigma},j}} \ \sup_{V\in A_{\sigma,\tilde{\sigma}}} | V(\zeta) |,$$

$$\phi_j(r) = \frac{r}{\log(e + B_j(r))}, \ r > 0.$$

The function $B_j(r)$, $r \in R_+$, is well defined because the sets $A_{\sigma,\tilde{\sigma}}$ and $\{\zeta \in R^n + iK_{\tilde{\sigma},j} ; | \zeta | = r)$ are compact and σ, $\tilde{\sigma}$ belong to the finite set. Moreover, for every $\varepsilon > 0$ there exists $C_{j,\varepsilon} > 0$ such that $B_j(r) \leq C_{j,\varepsilon} e^{\varepsilon r}$, $r \in R_+$, $j \in N$. Also, $\phi_j(r) > 0$ and $\phi_j(r) \to \infty$ as $r \to \infty$. Let $\{\varphi_j(r) = \max(1, \inf_{s\geq r} \phi_j(s)) ; j \in N\}$. We can replace the sequence in (1) by a unique function ϕ, with the property $Cr^{1/2} < \phi(r)$, as it is done in [1]. Now,

$$(2) \quad | R_{h_\nu,\sigma,\tilde{\sigma}}(\zeta) | \leq C_j' \exp(| \zeta | / \phi(| \zeta |)), \quad \zeta \in R^n + iK_{\tilde{\sigma},j}, \ j \in N.$$

Note that ϕ does not depend on σ, $\tilde{\sigma} \in \Lambda$, $h_\nu \in \Gamma$, $\nu \in N$ and $j \in N$. There exists an elliptic local operator $J(D)$ whose Fourier transform $J(\zeta)$ satisfies the estimate

$$(3) \quad | J(\zeta) | \geq C \exp\left(\frac{| \zeta |}{\phi(| \zeta |)}\right), | \eta | \leq \frac{| \xi |}{\sqrt{3}} + 1, \xi \in R^n,$$

and $J(\zeta)$ is an entire function of infraexponential growth in ζ. (Lemma 1.2 in [2]).

The functions $R_{h_\nu,\sigma,\tilde\sigma}(\zeta)/J^2(\zeta)$, $\zeta \in R^n - iI_{\tilde\sigma}$, $h_\nu \in \Gamma$, $\nu \in N$, σ, $\tilde\sigma \in \Lambda$, are holomorphic and (2) and (3) imply

(4) $$|R_{h_\nu,\sigma,\tilde\sigma}(\zeta)/J^2(\zeta)| \le C_j' \exp(-\sqrt{|\zeta|}), \quad \zeta \in R^n + iK_{\tilde\sigma,j},\ J \in N.$$

Let $\eta \in -I_{\tilde\sigma}$, $h_\nu \in \Gamma$, $\nu \in N$. Define

(5) $$H_{h_\nu,\sigma,\tilde\sigma}(z) = \frac{1}{(2\pi)^n}\int_{Im\zeta=\eta} e^{iz\zeta} R_{h_\nu,\sigma,\tilde\sigma}(\zeta)/J^2(\zeta)d\xi, \quad z \in R^n + iI_\sigma.$$

By the definition of the Fourier transform, it follows that $H_{h,\sigma,\tilde\sigma} \in \tilde{\mathcal{O}}(D^n + iI_\sigma)$ and decreases exponentially along the real axis outside the closed cone $\Gamma^0_{\tilde\sigma}$.

Let $\eta \in K_{\tilde\sigma,j}$, $|\eta| \le 1$, $h_\nu \in \Gamma$, $\nu \in N$, $p \in N_0^n$. Then (4) and (5) imply

$$|H^{(p)}_{h_\nu,\sigma,\tilde\sigma}(z)| \le C_j e^{-x\eta}\int_{R^n} e^{-y\xi}e^{-\sqrt{|\zeta|}}|\zeta|^{|p|}d\xi, \quad z \in R^n + iI_\sigma.$$

This implies that for every $h_\nu \in \Gamma$, $\nu \in N$, σ, $\tilde\sigma \in \Lambda$ and $p \in N_0^n$, $H^{(p)}_{h_\nu,\sigma,\tilde\sigma}(z)$, $z \in R^n + iI_\sigma$ is continuable to a continuous function $H^{(p)}_{h_\nu,\sigma,\tilde\sigma}(x)$ up to the real axis. Moreover, this function satisfies $|H_{h_\nu,\sigma,\tilde\sigma}(x)| \le C_j \exp(|x|/j)$, $j \in N$ (it is slowly increasing) and belongs to $\mathcal{E}(R^n)$. By the properties of $R_{h_\nu,\sigma,\tilde\sigma}(\zeta)$ it follows that for $h_\nu \in \Gamma$

$$H_{h_\nu,\sigma,\tilde\sigma}(x) \to H_{\sigma,\tilde\sigma}(x) = \frac{1}{(2\pi)^n}\int_{Im\zeta=\eta} e^{iz\zeta} R_{\sigma,\tilde\sigma}(\zeta)/J^2(\zeta)d\xi \ \ as\ \|h_\nu\| \to \infty,\ \nu \to \infty,$$

in $\mathcal{E}(R^n)$ and $H_{\sigma,\tilde\sigma}(x)$ is also slowly increasing. Also, for every ε

$$\sup_{x \in R^n} \|H_{h_\nu,\sigma,\tilde\sigma}(x) - H_{\sigma,\tilde\sigma}(x)\| e^{-\varepsilon|x|} \to 0,\ \nu \to \infty.$$

By Carleman's Lemma ([3] p. 395) $H_{h_\nu,\sigma,\tilde\sigma}(x + i\Gamma_\sigma 0)$ and $H_{\sigma,\tilde\sigma}(x + i\Gamma_\sigma 0)$ define Fourier hyperfunctions $\ell H_{h_\nu,\sigma,\tilde\sigma}(x)$ and $\ell H_{\sigma,\tilde\sigma}(x)$, respectively.

Hence

$$J^2(D)\sum_{\sigma\in\Lambda}\sum_{\tilde\sigma\in\Lambda} H_{h_\nu,\sigma,\tilde\sigma}(x + i\Gamma_\sigma 0) = \sum_\sigma F_{h_\nu,\sigma}(x + i\Gamma_\sigma 0) \cong f_{h_\nu}(x),\ h_\nu \in \Gamma,$$

$$J^2(D)\sum_{\sigma\in\Lambda}\sum_{\tilde\sigma\in\Lambda} H_{\sigma,\tilde\sigma}(x + i\Gamma_\sigma 0) = \sum_\sigma F_\sigma(x + i\Gamma_\sigma 0) \cong f(x).$$

Put

$$\sum_{\sigma\in\Lambda}\sum_{\tilde\sigma\in\Lambda} H_{h_\nu,\sigma,\tilde\sigma}(x) = q_{h_\nu}(x),\ h_\nu \in \Gamma \quad \text{and} \quad \sum_\sigma\sum_{\tilde\sigma} H_{\sigma,\tilde\sigma}(x) = q(x).$$

Since $H_{h_\nu,\sigma} = (\text{sgn}\tilde\sigma\ H_{h_\nu,\sigma,\tilde\sigma})$ converges in $\tilde{\mathcal{O}}((D^n + iU) \# D^n)$ to $H_\sigma = (\text{sgn}\tilde\sigma H_{\sigma,\tilde\sigma})$, as $\|h_\nu\| \to \infty$, $h_\nu \in \Gamma$, $\nu \to \infty$, for every $\sigma \in \Lambda$, Proposition 2 and the continuity of a local operator imply $f_{h_\nu} = J^2(D)(\ell q_{h_\nu})$, $h_\nu \in \Gamma$, $\nu \in N$ and $f = J^2(D)\ell q$.

Properties of q_{h_ν} and q follow from the corresponding properties of $H_{h_\nu,\sigma,\tilde\sigma}$ and $H_{\sigma,\tilde\sigma}$.

Remark. As we mentioned at the beginning of the proof of Theorem 1, we have that

$$\tilde F_h \text{ converges to } \tilde F,\ \|h\| \to \infty,\ h \in \Gamma \text{ in } \tilde{\mathcal{O}}((D^n + iI) \# D^n).$$

If we assume that there exists h_0 and a closed cone $\Gamma_1 \subset \Gamma$ such that the mapping

$$\Gamma_1 \cap \{h;\|h\| \ge h_0\} \to \tilde{\mathcal{O}}((D^n + iI) \# D^n),\ h \to \tilde F_h,$$

is continuous, then $A_{\sigma,\tilde\sigma} = \{R_{\sigma,\tilde\sigma},\ R_{h,\sigma,\tilde\sigma};\ h \in \Gamma_1,\ \|h\| \ge h_0\}$ is a compact set. In this case we have that there exists a unique elliptic local operator $J(D)$ and a family of functions $\{q_h;\ h \in \Gamma_1,\ \|h\| \ge h_0\}$ with the properties given in a), b), and c) for $h \in \Gamma_1$, $\|h\| \ge h_0$.

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