L^p-analysis on Homogeneous Manifolds of Reductive Type and Representation Theory

By Toshiyuki KOBAYASHI

Department of Mathematics, University of Tokyo

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§1. Introduction.

1.1. Let G be a real reductive linear Lie group, K a maximal compact subgroup of G, and θ the corresponding Cartan involution. Suppose that H is a closed θ-stable subgroup of G with finitely many connected components. Then the coset space G/H is called a *homogeneous manifold of reductive type*. Riemannian symmetric spaces (e.g. $GL(n, \boldsymbol{R})/O(n)$), semisimple Lie group manifolds (e.g. $GL(n, \boldsymbol{R})/\{e\} \simeq GL(n, \boldsymbol{R}) \times GL(n, \boldsymbol{R})/\mathrm{diag}\, GL(n, \boldsymbol{R})$), or more generally, reductive symmetric spaces (e.g. $GL(n, \boldsymbol{R})/O(p, n-p)$), and semisimple orbits in semisimple Lie algebras under the adjoint action (e.g. $GL(n, \boldsymbol{R})/GL(n_1, \boldsymbol{R}) \times \cdots \times GL(n_k, \boldsymbol{R})$ $(\sum n_j = n)$) are typical examples of homogeneous manifolds of reductive type. Various geometric structures of homogeneous manifolds of reductive type may be found in the survey [7] and references therein.

1.2. If G/H is of reductive type, then there exists a G-invariant measure $d\mu$ on G/H, which is unique up to a scalar multiple. Then we have a continuous representation of G on the Banach space $L^p(G/H; d\mu) \equiv L^p(G/H)$ $(p \geq 1)$ by left translations. A fundamental problem in L^p-analysis on a homogeneous manifold G/H is to construct an irreducible representation of G in a closed G-invariant subspace of $L^p(G/H)$. In particular, if $p = 2$, then G acts unitarily on the Hilbert space $L^2(G/H)$ and an irreducible representation π of G is called a *discrete series representation* for G/H, provided π is realized in a closed G-invariant subspace of $L^2(G/H)$. Discrete series representations are automatically unitary and we denote by $\mathrm{Disc}(G/H)$ the unitary equivalence classes of discrete series representations for G/H. By definition, $\mathrm{Disc}(G/H) \subset \widehat{G}$, where $\widehat{G}$ denotes the unitary dual of G.

1.3. It is a celebrated work due to Harish-Chandra that $\mathrm{Disc}(G/\{e\}) \neq \emptyset$ iff $\mathrm{rank}\, G = \mathrm{rank}\, K$. Generalizing this, Flensted-Jensen, Matsuki and Oshima proved in [1] and [12] that $\mathrm{Disc}(G/H) \neq \emptyset$ iff

(1.3.1) $\quad \mathrm{rank}\, G/H = \mathrm{rank}\, K/H \cap K$

for a reductive symmetric space G/H (see §3.2 for definition).

1.4. However, except for reductive symmetric spaces, our current knowledge about the existence of discrete series representations for G/H (or more generally, the existence of irreducible representations in $L^p(G/H)$) is very poor. This is partly because the known methods relied on

i) the commutativity of G- invariant differential operators on G/H,
ii) a Cartan decomposition $G = KAH$ ([1], §2),
iii) the dual space G^d/H^d (see [2] for the notation).

In our more general setting where G/H is of reductive type, (i) may fail, and neither (ii) nor (iii) always exists.

Thus, we need a new method to investigate the L^p-analysis on G/H for a general homogeneous manifold of reductive type. Our approach in this paper is based on the recent theory of the discrete decomposable restriction of unitary representations ([5] and [6]), and on a comparison theorem of two homogeneous manifolds (Theorem 2.7) together with well-developed results on reductive symmetric spaces ([1], [2], and [12]).

§2. Invariant measure on homogeneous manifolds of reductive type.

2.1. For a reductive symmetric space G/H, there is a generalized Cartan decomposition "$G = KAH$" with $A \simeq \boldsymbol{R}^l$, which is an analog of the polar coordinate in the Euclidean space. Then, the corresponding integration formula ([1], Theorem 2.6) is a basic tool on harmonic analysis on a symmetric space G/H, because the L^p-estimate of functions on G/H can be studied

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by the decay condition along the abelian part $A \simeq \boldsymbol{R}^l$.

Unfortunately, there does not always exist an analog of a generalized Cartan decomposition $G = KAH$ in our general setting where G/H is of reductive type, as one can easily observe by an argument of dimension (e.g. the case where $\dim H$ is very small).

Thus, in order to give an L^p-estimate of functions on G/H, we need a nice estimate of the invariant measure on G/H without using the KAH decomposition, which is the main goal of this section.

Let us fix some notation. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G corresponding to a Cartan involution θ of G, and we fix an $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}$ such that $B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite, $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite and that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal to each other (e.g. we can take B to be the Killing form if $\mathfrak{g}$ is semisimple). We write $\|X\|$ $(X \in \mathfrak{p})$ for the induced norm on $\mathfrak{p}$.

2.2. Suppose G/H is a homogeneous manifold of reductive type. We write $\mathfrak{h}$ for the Lie algebra of H. Then the restriction $B|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate. Let $\mathfrak{h}^\perp$ be the orthogonal complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$. Then, we have a direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^\perp$.

Theorem 2.2. *Let G/H be a homogeneous manifold of reductive type. Then there exists a non-negative function $\delta : \mathfrak{h}^\perp \cap \mathfrak{p} \to \boldsymbol{R}$ such that*

$$\int_{G/H} f(x)\,d\mu(x) = \int_K \int_{\mathfrak{h}^\perp \cap \mathfrak{p}} f(ke^X \cdot H)\,\delta(X)\,dk\,dX$$

for any $f \in C_c(G/H)$.

Here dk is the bi-invariant measure on K and dX is the Lebesgue measure on $\mathfrak{h}^\perp \cap \mathfrak{p}$. Furthermore, there exist constants $\nu_{G/H} > 0$ and $C > 0$ such that

$$\delta(X) \le C \exp(\nu_{G/H} \|X\|) \text{ for any } X \in \mathfrak{h}^\perp \cap \mathfrak{p}.$$

Remark 2.3. One can prove that there exists a constant $\nu \equiv \nu(G)$ such that $\nu_{G/H} \le \nu$ for any θ-stable closed subgroup H with finitely many connected components.

2.4. Given $\xi \in \boldsymbol{R}$, we introduce a subspace of continuous functions on G/H:

$$C(G/H\,;\xi) := \{f \in C(G/H) : \sup_{k\in K}\ \sup_{X\in\mathfrak{p}\cap\mathfrak{h}^\perp} f(k\exp X)\exp(\xi\|X\|) < \infty\}.$$

There is an obvious inclusive relation $C(G/H\,;\xi) \subset C(G/H\,;\xi')$ for $\xi > \xi'$.

Corollary 2.4. *Let G/H be a homogeneous manifold of reductive type. If $1 \le p \le \infty$, then we have*

$$C(G/H\,;\xi) \subset L^p(G/H) \text{ if } p\,\xi > \nu_{G/H}.$$

We put $C^\infty(G/H\,;\xi) := C(G/H\,;\xi) \cap C^\infty(G/H)$. We say that an admissible irreducible representation π of G is *of decay type ξ for G/H* if the underlying $(\mathfrak{g}, K)$-module π_K can be realized as a subrepresentation of $C^\infty(G/H\,;\xi)$. If π is of decay type ξ for G/H with $\xi > \frac{1}{2}\nu_{G/H}$, then π_K is unitarizable and $\mathrm{Disc}(G/H) \ne \emptyset$ by Corollary 2.4.

2.5. Next, we give a comparison theorem of invariant measures of two homogeneous manifolds $G'/H' \subset G/H$. Consider the following setting: G is a real reductive linear Lie group with a Cartan involution θ; both H and G' are θ-stable closed subgroups of G with finitely many connected components. Let $H' := H \cap G'$. Then G'/H' is also of reductive type. We write $\iota : G'/H' \hookrightarrow G/H$ for the natural embedding, and $\iota^* : C(G/H) \to C(G'/H')$ for the pullback of continuous functions.

2.6. In the setting of §2.5, we write $\mathfrak{g}'$ and $\mathfrak{h}'$ for the Lie algebras of G' and H', respectively. We put $\mathfrak{p}' := \mathfrak{g}' \cap \mathfrak{p}$, and define $\mathfrak{h}'^\perp$ to be the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{g}'$ with respect to $B|_{\mathfrak{g}'\times\mathfrak{g}'}$. Then, we have direct sum decompositions $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{h}'^\perp$ and $\mathfrak{p}' = (\mathfrak{h}' \cap \mathfrak{p}') \oplus (\mathfrak{h}'^\perp \cap \mathfrak{p}')$. We define

(2.6.1) $b(G'/H'\,; G/H) := \sin\phi(\mathfrak{q}' \cap \mathfrak{p}', \mathfrak{h} \cap \mathfrak{p})$.

Here, $(0 \le)\ \phi(\mathfrak{q}' \cap \mathfrak{p}', \mathfrak{h} \cap \mathfrak{p})\ \left(\le \frac{\pi}{2}\right)$ is the angle between the subspaces of $\mathfrak{p}$, $\mathfrak{q}' \cap \mathfrak{p}'$ and $\mathfrak{h} \cap \mathfrak{p}$.

Lemma 2.6. *If G'/H' is noncompact, then $b(G'/H'\,; G/H) > 0$.*

2.7. Here is a comparison theorem that gives an estimate of the decay of a function on G/H restricted to a submanifold G'/H'.

Theorem 2.7. *Let $b := b(G'/H'\,; G/H)$ be the constant given in* (2.6.1). *Then*

$$\iota^* C(G/H\,;\xi) \subset C(G'/H'\,; b\xi) \quad \text{for any } \xi \ge 0.$$

Corollary 2.8. *Retain the setting of Theorem 2.7. Let $\nu_{G'/H'}$ be the constant given in Theorem 2.2. Then for any $1 \le p \le \infty$, we have*

$$\iota^* C(G/H\,;\xi) \subset L^p(G'/H') \quad \text{if } bp\xi > \nu_{G'/H'}.$$

§3. Irreducible representations in $L^p(G/H)$.

In this section, we investigate a sufficient condition for the existence of irreducible representations realized in closed subspaces of $L^p(G/H)$. In particular, we construct new discrete series representations which play a fundamental role in the non-commutative harmonic analysis on G/H.

3.1. Suppose $G' \subset G$ are real reductive Lie groups with maximal compact groups $K' \subset K$. We say that $\pi \in \hat{G}$ is *K'-admissible* if

$$\dim \mathrm{Hom}_{K'}(\tau, \pi|_{K'}) < \infty \quad \text{for any } \tau \in \widehat{K'}.$$

Then we also say that the underlying $(\mathfrak{g}_C, K)$-module π_K is *K'-admissible.* Suppose we are in the setting 2.5 and 2.6. Let $b = b(G'/H'; G/H)$. Here is a key lemma in the proof of Theorem 3.5, the main result in this section:

Lemma 3.1. *Let $\pi \in \hat{G}$ be K'-admissible. If π is of decay type ξ for G/H, then there exists $\tau \in \widehat{G'}$ that is of decay type $b\xi$ for G'/H'.*

This lemma follows directly from an algebraic result in representation theory ([5], Part III, Proposition 1.6).

3.2. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. We fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Let τ be an involutive automorphism of G, and G' an open subgroup of $G^\tau := \{g \in G : \tau g = g\}$. Then (G, G') is called a *reductive symmetric pair.* The homogeneous manifold G/G' is called a *reductive symmetric space* (or a *semisimple symmetric space* if G is semisimple). We say that τ is *in a standard position* with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$ if the following four conditions are satisfied:

(3.2.1) $\tau\theta = \theta\tau$.

(3.2.2) $\tau(\mathfrak{t}) = \mathfrak{t}$.

(3.2.3) $\mathfrak{t}^{-\tau}$ is a maximal abelian subspace in $\mathfrak{k}^{-\tau}$.

(3.2.4) $\{\alpha|_{\mathfrak{t}^{-\tau}} : \alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t})\} \setminus \{0\}$ defines a positive system $\Sigma^+(\mathfrak{k}, \mathfrak{t}^{-\tau})$ of $\Sigma(\mathfrak{k}, \mathfrak{t}^{-\tau})$.

Here, we wrote $\mathfrak{m}^{-\tau} := \{X \in \mathfrak{m} : \tau X = -X\}$ for a subspace $\mathfrak{m}$ of $\mathfrak{g}$.

3.3. Given an element $X \in \sqrt{-1}\,\mathfrak{t}$, we define a θ-stable parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l}_C + \mathfrak{u} \equiv \mathfrak{l}(X)_C + \mathfrak{u}(X) \ \ (\subset \mathfrak{g}_C)$$

such that $\mathfrak{l}_C$ and $\mathfrak{u}$ are the sum of eigenspaces with 0 and positive eigenvalues of $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g}_C)$, respectively. Then $\mathfrak{l}_C$ is the complexification of the Lie algebra of $L = Z_G(X)$, the centralizer of X in G. We write $A_\mathfrak{q}(\lambda)$ for the Zuckerman's derived functor $(\mathfrak{g}, K)$-module for a metaplectic $L^\sim$-character C_λ in the good range (see [3]).

We say that $\mathfrak{q}$ is *in a standard position* for a fixed positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ if X sits inside a dominant chamber with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$. We denote by $\boldsymbol{R}_+\langle \mathfrak{u} \cap \mathfrak{p}_C\rangle$ the cone in $\sqrt{-1}\,\mathfrak{t}^*$ defined by the $\boldsymbol{R}_+$-span of $\Delta(\mathfrak{u} \cap \mathfrak{p}_C, \mathfrak{t})$.

Fact 3.3. *Suppose (G, G') is a reductive symmetric pair defined by an involution τ in a standard position with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Suppose $\mathfrak{q}$ is a θ-stable parabolic subalgebra in a standard position with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Then the following three conditions on $(\mathfrak{q}, G, G')$ are equivalent*:

1) $\boldsymbol{R}_+\langle \mathfrak{u} \cap \mathfrak{p}_C\rangle \cap \sqrt{-1}(\mathfrak{t}^{-\tau})^* = \{0\}$.

2) *The restriction of $A_\mathfrak{q}(\lambda)$ to K' is K'-admissible.*

3) *$A_\mathfrak{q}(\lambda)$ is decomposed into an algebraic sum of irreducible $(\mathfrak{g}', K')$-modules.*

Proof. See [5], Part I, Theorem 3.2 for (1) $\Rightarrow$ (2); and [5], Part III, Theorem 4.2 for other implications. □

3.4. Let (G, H) and (G, G') be reductive symmetric pairs defined by ivolutive automorphisms σ and τ of G, respectively, which are in a standard position with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$. We employ an analogous notation of §3.2 for σ. If (1.3.1) is satisfied, then $\mathfrak{t}^{-\sigma}$ is a maximal abelian subspace in $\mathfrak{g}^{-\sigma}$. We denote by $W(\mathfrak{g}, \mathfrak{t}^{-\sigma}) \supset W(\mathfrak{k}, \mathfrak{t}^{-\sigma})$ the Weyl groups of $\Sigma(\mathfrak{g}, \mathfrak{t}^{-\sigma}) \supset \Sigma(\mathfrak{k}, \mathfrak{t}^{-\sigma})$, respectively. Fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{t}^{-\sigma})$ which contains $\Sigma^+(\mathfrak{k}, \mathfrak{t}^{-\sigma})$. A θ-stable parabolic subalgebra $\mathfrak{q} \equiv \mathfrak{q}(X) = \mathfrak{l}_C + \mathfrak{u}$ is attached by a strictly dominant element $X \in \sqrt{-1}\,\mathfrak{t}^{-\sigma}$ with respect to $\Sigma^+(\mathfrak{g}, \mathfrak{t}^{-\sigma})$ (see §3.3). For each $w \in W(\mathfrak{k}, \mathfrak{t}^{-\sigma}) \setminus W(\mathfrak{g}, \mathfrak{t}^{-\sigma})$, we choose a representative $m_w \in K$ such that $\mathrm{Ad}(m_w)X$ is dominant with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$, and define a θ-stable parabolic subalgebra $\mathfrak{q}^w := \mathrm{Ad}(m_w)\mathfrak{q} = \mathfrak{l}_C + \mathfrak{u}^w$, where $\mathfrak{u}^w := \mathrm{Ad}(m_w)\mathfrak{u}$.

3.5. We write $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} : \mathfrak{g}_C^* \to \mathfrak{g}'^*_C$ for the natural projection. We denote by $\mathrm{Ass}(\pi) \subset \mathfrak{g}'^*_C$ the associated variety of a $(\mathfrak{g}'_C, K')$-module π of finite length.

Theorem 3.5. *Let G/H be a reductive symmetric space satisfying* (1.3.1). *Retain the notation in* §3.4. *We assume that there exists $w \in W(\mathfrak{k}, \mathfrak{t}^{-\sigma}) \setminus W(\mathfrak{g}, \mathfrak{t}^{-\sigma})$ such that*

$$\boldsymbol{R}_+\langle \mathfrak{u}^w \cap \mathfrak{p}_C\rangle \cap \sqrt{-1}(\mathfrak{t}^{-\tau})^* = \{0\}.$$

We put $H'_x := G' \cap xHx^{-1}$ for $x \in K$. Then the following statements hold:

1) *For any $x \in K$, there exists an irreducible $(\mathfrak{g}'_C, K')$-module π such that*

(3.5.1) $\mathrm{Ass}(\pi) \supset \mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}\mathrm{Ad}(K_C)(\mathfrak{u}^w \cap \mathfrak{p}_C)$,

(3.5.2) $\mathrm{Hom}_{(\mathfrak{g}'_C, K')}(\pi, C^\infty(G/H) \cap \bigcap_{1\le p\le\infty} L^p(G'/H'_x)) \neq \{0\}$.

In particular, $\mathrm{Disc}(G'/H'_x) \neq \emptyset$ *for any* $x \in K$.
2) *Assume moreover that* $Z_G(\mathfrak{t}^{-\sigma})$ *is compact. Then* $\mathrm{Disc}(G'/H'_x) \cap \mathrm{Disc}(G') \neq \emptyset$ *for any* $x \in K$.

The point here is that G'/H'_x gives different homogeneous manifolds of reductive type as $x \in K$ varies. In general, G'/H'_x is non-symmetric. A recent study of the double coset decomposition $G^\tau \backslash G/G^\sigma$ by Matsuki ([10] and [11]) helps us to compute explicitly the isotropy subgroup H'_x.

Example 3.6. The assumptions of Theorem 3.5 are satisfied, if the triple of Lie groups $(G, H, G') \equiv (G, G^\sigma, G^\tau)$ is one of the following cases:

$(U(2p, 2q),\ Sp(p, q),\ U(i, j) \times U(2p-i, 2q-j))$,
$(O(p, q),\ O(m) \times O(p-m, q),\ O(p, q-r) \times O(r))$,
$(U(p, q),\ U(m) \times U(p-m, q),\ U(p, q-r) \times U(r))$,
$(Sp(p, q),\ Sp(m) \times Sp(p-m, q),\ Sp(p, q-r) \times Sp(r))$,

where $0 \leq i \leq 2p$, $0 \leq j \leq 2p$, $0 \leq 2m \leq p$ and $0 \leq r \leq q$. We note that σ does not commute with τ if i or j is odd in the first case.

Remark 3.7. The special case where $\dim H + \dim G' = \dim G + \dim(H \cap G')$ (and $x = e$) was studied in [5], Corollary 5.6, where we dealt with so-called the (non-symmetric) *spherical homogeneous spaces.*

Example 3.8. The homogeneous manifolds

$$G/H = O(4m, n)/U(2m, j),\ (0 \leq 2j \leq n)$$

admit discrete series representations. This was previously known when $n = 2j$ and $2j - 1$, where G/H is a semisimple symmetric space, and a non-symmetric spherical homogeneous space, respectively. Other cases are new.

§4. Holomorphic discrete series representations.

4.1. In this section, we investigate a nice subset of $\mathrm{Disc}(G/H)$, namely, "holomorphic discrete series representations for G/H". As in Theorem 3.4, analogous results in this section hold for L^p-representations $(1 \leq p \leq \infty)$, but we restrict to the case of $p = 2$ for simplicity.

We assume that G/K is an irreducible Hermitian symmetric space. Equivalently, G is simple and the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is one dimensional. Then we can take $Z \in \mathfrak{c}(\mathfrak{k})$ so that

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

are $0, \sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $\mathrm{ad}\, Z$. For $\pi \in \hat{G}$. we say that π is a *highest weight module* if there exists a non-zero vector in the underlying $(\mathfrak{g}, K)$-module annihilated by $\mathfrak{p}^+$. We denote by $\hat{G}_{h.w.}$ $(\subset \hat{G})$ the unitary equivalence class of irreducible unitary highest weight modules. Then an element of $\mathrm{Disc}(G/H) \cap \hat{G}_{h.w.}$ is called a *holomorphic discrete series representation for* G/H. This terminology coincides with the usual one if $H = \{e\}$. Lowest weight modules and anti-holomorphic discrete series representations are defined similarly with $\mathfrak{p}^+$ replaced by $\mathfrak{p}^-$.

Suppose τ is an involutive automorphism of G commuting with θ. Since $\tau\mathfrak{c}(\mathfrak{k}) = \mathfrak{c}(\mathfrak{k})$, there are two exclusive possibilities:

(4.1.1) $\tau Z = Z,$

(4.1.2) $\tau Z = -Z.$

We note that the induced action of τ on G/K is holomorphic for (4.1.1); anti-holomorphic for (4.1.2).

4.2. Retain the setting of §4.1. Let σ be an involutive automorphism of G satisfying $\sigma\theta = \theta\sigma$ and $\sigma Z = -Z$ (see (4.1.2)), and $x \in K$. We consider the following two settings:

Setting 1: $L_x := \{g \in G : x\sigma(g)x^{-1} = g\}$.

Setting 2: Let τ be an involutive automorphism of G satisfying (4.1.1) and $G' := G^\tau$. We put $H'_x := G^\tau \cap xG^\sigma x^{-1}$.

Here is an existence theorem of holomorphic discrete series representations for homogeneous manifolds of reductive type.

Theorem 4.2. *For any* $x \in K$, *we have*

$\#(\mathrm{Disc}(G/L_x) \cap \mathrm{Disc}(G) \cap \hat{G}_{h.w.}) = \infty$ *in Setting* 1,

$\#(\mathrm{Disc}(G'/H'_x) \cap \mathrm{Disc}(G') \cap \widehat{G'}_{h.w.}) = \infty$ *in Setting* 2.

4.3. A very special case (i.e. $x = e$ in Setting 1) leads to a new and elementary proof of the following result due to Ólafsson and Ørsted:

Corollary 4.3 (see [13]). *There exist* (*infinitely many*) *holomorphic discrete series representations for a psend-Riemannian symmetric space of Hermitian type.*

4.4. Choosing $x \in K$, τ and σ, we can obtain a number of new holomorphic discrete series representations for homogeneous manifolds of reductive type $(G/L_x$ and $G'/H'_x)$. For instance, we have:

Example 4.4. The homogeneous manifolds

$$G/H = Sp(2n, \mathbf{R})/(Sp(n_0, \mathbf{C}) \times GL(n_1, \mathbf{C}) \times \cdots \times GL(n_k, \mathbf{C}))\ (\sum n_j = n)$$

admit holomorphic discrete series representations. We note that G/H is a semisimple symmetric space if and only if $n_1 = n_2 = \cdots = n_k = 0$, which is previously the known case.

A detailed proof will appear in [8] and [9].

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