Periodic Solutions of the Heat Convection Equations in Exterior Domains

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1. Introduction. Let $\Omega = K^c \subset \boldsymbol{R}^3$ where K is a compact set whose boundary ∂K is of class C^2. We put $\partial\Omega = \Gamma = \partial K$, $\hat{\Gamma} = \Gamma \times (0, \infty)$ and $\hat{\Omega} = \Omega \times (0, \infty)$. Then we consider the periodic problem for the heat convection equation (HCE):

$$(1)\quad \begin{cases} u_t + (u \cdot \nabla) u = -(\nabla p)/\rho + \{1 - \alpha(\theta - \Theta_0)\} g + \nu \Delta u & \text{in } \hat{\Omega}, \\ \operatorname{div} u = 0 & \text{in } \hat{\Omega}, \\ \theta_t + (u \cdot \nabla)\theta = \kappa \Delta \theta & \text{in } \hat{\Omega}, \end{cases}$$

$$(2)\quad u(x, t)|_{\hat{\Gamma}} = 0,\ \theta(x, t)|_{\hat{\Gamma}} = \chi(x, t)\ (> 0), \quad \lim_{|x| \to \infty} u(x) = 0,\ \lim_{|x| \to \infty} \theta(x) = 0, \text{ for } t > 0,$$

$$(3)\quad u(\cdot, T) = u(\cdot, 0),\ \theta(\cdot, T) = \theta(\cdot, 0).$$

Here $u = u(x)$ is the velocity vector, $p = p(x)$ is the pressure and $\theta = \theta(x)$ is the temperature; ν, κ, α, ρ and $g = g(x)$ are the kinematic viscosity, the thermal conductivity, the coefficient of volume expansion, the density at $\theta = \Theta_0$ and the gravitational vector, respectively. As for the exterior problem of (HCE), Hishida [2] showed the global existence of the strong solution for the initial value problem (IVP) in the case that K is a ball. Recently, Ōeda-Matsuda [7] showed the existence and uniqueness of weak solutions of (IVP) when K is a compact set with the boundary of class C^2. Moreover, Ōeda [10] obtained the stationary weak solutions for the similar exterior domain to that of [7]. In [7] and [10], we used "the extending domain method" to get weak solutions. Namely, it is expected that the exterior domain Ω can be approximated by interior domains $\Omega_n = B_n \cap \Omega$ (B_n is a ball with radius n and center at O) as $n \to \infty$ (see Ladyzhenskaya [3]). The purpose of the present paper is to show the existence of periodic weak solutions of (HCE) by using "the extending domain method".

2. Preliminaries. We make several assumptions: (A1) $\omega_0 \subset \operatorname{int} K$ (ω_0 being a neighbourhood of the origine O) and $K \subset B = B(O, d)$, where B is a ball with radius d and center at O. (A2) $\partial\Omega = \Gamma = \partial K \in C^2$. (A3) $g(x)$ is a bounded and continuous vector function in $\boldsymbol{R}^3 \setminus \omega_0$. Moreover there exist $R_0 > 0$, $C_{R_0} > 0$ such that $|g| \le C_{R_0}/|x|^{\frac{5}{2}+\varepsilon}$ for $|x| \ge R_0$ ($\varepsilon > 0$ is arbitrary). (A4) $\chi \in C^2(\Gamma \times [0, \infty))$ and is periodic with respect to t with period T.

Remark 1. Thanks to (A3), we see $g \in L^p(\Omega)$ for $p \ge \dfrac{6}{5}$.

We prepare a lemma which gives us an auxiliary function (see [1] p. 131 and [11] p.175):

Lemma 2.1. *There is a function $\bar{\theta}(x, t)$ which possesses the following properties* (i) ~ (iv): (i) $\bar{\theta} = \chi$ *on* $\hat{\Gamma}$. (ii) $\bar{\theta}(x, t) \in C_0^2(\boldsymbol{R}_x^3)$ *for any fixed t and θ, θ_t are continuous for $t \in [0, T]$.* (iii) *$\bar{\theta}$ is periodic in t with period T.* (iv) *For any $\varepsilon > 0$ and $p > 1$, we can retake $\bar{\theta}$, if necessary, such that* $\sup_{t \in [0,T]} \| \bar{\theta}(t) \|_{L^p} < \varepsilon$.

Now we make a change of variable: $\theta = \hat{\theta} + \bar{\theta}$, and after changing of variable, we use the same letter θ. Equations (1), (2), and (3) are transformed to the following:

$$(4)\quad \begin{cases} u_t + (u \cdot \nabla) u = -(\nabla p)/\rho - \alpha\theta g + \nu \Delta u + \{1 - \alpha(\bar{\theta} - \Theta_0)\} g & \text{in } \hat{\Omega}, \\ \operatorname{div} u = 0 & \text{in } \hat{\Omega}, \\ \theta_t + (u \cdot \nabla)\theta = \kappa \Delta \theta - (u \cdot \nabla)\bar{\theta} - \bar{\theta}_t + \kappa \Delta \bar{\theta} & \text{in } \hat{\Omega}, \end{cases}$$

$$(5)\quad u|_{\hat{\Gamma}} = 0,\ \theta|_{\hat{\Gamma}} = 0,\ \lim_{|x| \to \infty} u(x) = 0, \quad \lim_{|x| \to \infty} \theta(x) = 0,$$

$$(6)\quad u(\cdot, T) = u(\cdot, 0),\ \theta(\cdot, T) = \theta(\cdot, 0).$$

We put $G = \Omega$ or Ω_n, $\hat{G}' = G \times [0, T]$ and $\widehat{G \cup \Gamma}' = (G \cup \Gamma) \times [0, T]$. Then we write
$W^{k,p}(G) = \{u ; D^\alpha u \in L^p(G), |\alpha| \le k\}$, $W_0^{k,p}(G)$ = the completion of $C_0^k(G)$ in $W^{k,p}(G)$,
$D_\sigma(G) = \{\varphi \in C_0^\infty(G); \operatorname{div} \varphi = 0\}$, $D(G) = \{\phi \in C_0^\infty(G \cup \Gamma); \phi(\Gamma) = 0\}$,
$H_\sigma(G)$ (resp. $H_\sigma^1(G)$) = the completion of $D_\sigma(G)$ in $L^2(G)$ (resp. $W^{1,2}(G)$),
$H_0^1(\Omega_n)$ = the completion of $D(\Omega_n)$ in $W^{1,2}(\Omega_n)$ (it turns out $H_0^1(\Omega_n) = W_0^{1,2}(\Omega_n)$),
V(resp. W) = the completion of $D_\sigma(\Omega)$ (resp. $D(\Omega)$) in $\| \cdot \|_{N(\Omega)}$, where $\| u \|_{N(\Omega)} = \| \nabla u \|_{L^2(\Omega)}$,
$\hat{D}_\sigma(\hat{G}) = \{\varphi \in C_0^\infty(\hat{G}'); \operatorname{div} \varphi = 0\}$, $\hat{D}(\hat{G}) = \{\psi$

$\in C_0^\infty(\widehat{G \cup \Gamma'}); \varphi(\hat{\Gamma}) = 0\}$,
$\hat{D}_{\sigma,\pi}(\hat{G}) = \{\varphi \in C_\sigma(\hat{G}); \varphi(x, T) = \varphi(x, 0)\}$, $\hat{D}_\pi(\hat{G}) = \{\psi \in \hat{D}(\hat{G}); \psi(x, T) = \psi(x, 0)\}$,
$L^2_\pi(0, T; H^1_\sigma(\Omega_n)) = \{u \in L^2(0, T; H^1_\sigma(\Omega_n));$
$u(x, T) = u(x, 0)$ a.e. $x \in \Omega_n\}$,
$L^2_\pi(0, T; H^1_0(\Omega_n)) = \{\theta \in L^2(0, T; H^1_0(\Omega_n));$
$\theta(x, T) = \theta(x, 0)$ a.e. $x \in \Omega_n\}$,
$L^2_\pi(0, T; L^6(\Omega)) = \{f \in L^2(0, T; L^6(\Omega_n));$
$f(x, T) = f(x, 0)$ a.e. $x \in \Omega\}$.

We state some inequalities. (see Chap. I of [3]).

Lemma 2.2. *Assume the space dimension is 3. G is permitted unbounded. Then*

(i) *For $u \in W_0^{1,2}(G)$ (or V or W), we have*

(7) $\| u \|_{L^6(G)} \le c \| \nabla u \|_{L^2(G)}$, *where* $c = (48)^{1/6}$.

(ii) (Hölder's inequality) *If each integral makes sense, then we have*

$$(8)\qquad |((u\cdot\nabla)v, w)_G| \le 3^{\frac{1}{p}+\frac{1}{r}} \| u \|_{L^p(G)} \cdot \| \nabla v \|_{L^q(G)} \cdot \| w \|_{L^r(G)},$$

where $p, q, r > 0$ *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

We state another lemma (see [3]):

Lemma 2.3. (Friedrichs). *Suppose G is a bounded domain in $\boldsymbol{R}^n$ and its boundary ∂G is of class C^2. Let us take an orthonormal basis $\{w_k\}_{k=1}^\infty$ of $L^2(G)$. Then for any $\varepsilon > 0$, there exists a number N_ε such that*

$$(9)\qquad \| u \|^2_{L^2(G)} \le \sum_{k=1}^{N_\varepsilon} (u, w_k)^2 + \varepsilon \| u \|^2_{W^{1,m}(G)}$$

for all $u \in W_0^{1,m}(G)$,

where $m > \frac{2n}{n+2}$ $(n \ge 2)$, $m \ge 1$ $(n = 1)$ *and N_ε is independent of u.*

3. Results. We state the definition of a periodic weak solution.

Definition 3.1. ${}^t(u, \theta) \in (L^2(0, T; V) \cap L^2_\pi(0, T; L^6(\Omega))) \times (L^2(0, T; W) \cap L^2_\pi(0, T; L^6(\Omega)))$ *is called a periodic weak solution of* (HCE) *if it satisfies* (10) *and* (11):

$$(10)\qquad \int_0^T \{(u, \varphi_t) + ((u\cdot\nabla)\varphi, u) - \nu(\nabla u, \nabla\varphi) - (\alpha g\theta, \varphi) + ((1 - \alpha(\bar{\theta} - \Theta_0))g, \varphi)\}dt = 0,$$

$$(11)\qquad \int_0^T \{(\theta, \psi_t) + ((u\cdot\nabla)\psi, \theta) - \kappa(\nabla\theta, \nabla\psi) - ((u\cdot\nabla)\bar{\theta}, \psi) - (\bar{\theta}_t, \psi) - \kappa(\nabla\bar{\theta}, \nabla\psi)\}dt = 0,$$

for all $\varphi \in \hat{D}_{\sigma,\pi}(\hat{\Omega})$ *and* $\psi \in \hat{D}_\pi(\hat{\Omega})$.

Remark 2. Let $u \in V$, $\theta \in W$, then $u(\Gamma) = 0$, $\theta(\Gamma) = 0$ and from (i) of Lemma 2.2, $\lim_{|x|\to\infty} u(x) = 0$, $\lim_{|x|\to\infty} \theta(x) = 0$.

Then we mention a main theorem.

Theorem 3.2. *Suppose assumpions* (A1) $\sim$ (A4) *are satisfied. If* $3c^2\alpha \| g \|_{L^{\frac{3}{2}}(\Omega)} < \sqrt{\kappa\nu}$ *(where* $c = (48)^{\frac{1}{6}}$*), then a periodic weak solution of* (HCE) *exists.*

4. Proof of results. To construct a periodic weak solution, we use "the extending domain method". We first show a lemma by which we have periodic weak solutions of interior problems (P_n) in domains $\Omega_n = B_n \cap \Omega$. We state the interior problem (P_n):

$$(12)\qquad \begin{cases} v_t + (v\cdot\nabla)v = -(\nabla p)/\rho - \alpha\Theta g \\ \qquad + \nu\Delta v + \{1 - \alpha(\bar{\theta} - \Theta_0)\}g & \text{in } \hat{\Omega}_n, \\ \qquad \operatorname{div} v = 0 & \text{in } \hat{\Omega}_n, \\ \Theta_t + (v\cdot\nabla)\Theta = \kappa\Delta\Theta - (v\cdot\nabla)\bar{\theta} - \bar{\theta}_t \\ \qquad + \kappa\Delta\bar{\theta} & \text{in } \hat{\Omega}_n, \end{cases}$$

(13) $v|_{\partial\Omega_n} = 0$, $\Theta|_{\partial\Omega_n} = 0$, where $\partial\Omega_n = \Gamma + \partial B_n$,

(14) $u(\cdot, T) = u(\cdot, 0)$, $\Theta(\cdot, T) = \Theta(\cdot, 0)$.

The definition of a periodic weak solution for the problem (P_n) is as follows:

Definition 4.1. ${}^t(v, \Theta) \in (L^2_\pi(0, T; H^1_\sigma(\Omega_n))) \times (L^2_\pi(0, T; H^1_0(\Omega_n)))$ *is called a periodic weak solution for* (P_n) *if it satisfies the following*:

$$(15)\qquad \int_0^T \{(v, \varphi_t) + ((v\cdot\nabla)\varphi, v) - \nu(\nabla v, \nabla\varphi) - (\alpha g\Theta, \varphi) + ((1 - \alpha(\bar{\theta} - \Theta_0))g, \varphi)\}dt = 0,$$

$$(16)\qquad \int_0^T \{(\Theta, \psi_t) + ((v\cdot\nabla)\psi, \Theta) - \kappa(\nabla\Theta, \nabla\psi) - ((v\cdot\nabla)\bar{\theta}, \psi) - (\bar{\theta}_t, \psi) - \kappa(\nabla\bar{\theta}, \nabla\psi)\}dt = 0,$$

for $\varphi \in \hat{D}_{\sigma,\pi}(\hat{\Omega}_n)$ *and* $\psi \in \hat{D}_\pi(\hat{\Omega}_n)$.

Here we will present an important lemma to carry out "the extending domain method".

Lemma 4.2. *Suppose assumptions* (A1)$\sim$(A4) *are satisfied. Then there exists a satisfactory extension $\bar{\theta}$ which is independent of Ω_n such that, using it in common to all Ω_n, we can construct a periodic weak solution ${}^t(v_n, \Theta_n)$ of* (P_n).

Proof of Lemma 4.2. Let n be arbitrarily fixed. We use Galerkin's method. Let $\{w_j\} \subset D_\sigma(\Omega_n)$ (resp. $\{z_j\} \subset D(\Omega_n)$) be a sequence of functions, orthnormal in $L^2(\Omega_n)$ and total in $H^1_\sigma(\Omega_n)$ (resp. $H^1_0(\Omega_n)$). We put

$$(17)\qquad v^{(m)}(t) = \sum_{j=1}^m \alpha_{j,m}(t) w_j, \quad \Theta^{(m)}(t) = \sum_{j=1}^m \beta_{j,m}(t) z_j,$$

then we consider an initial value problem for the following ordinary differential equations:

$$(18)\qquad \frac{d}{dt}(v^{(m)}(t), w_j) + ((v^{(m)}\cdot\nabla)v^{(m)}, w_j) = -\nu(\nabla v^{(m)}, \nabla w_j) - (\alpha g\Theta^{(m)}, w_j) + (\{1 - \alpha(\bar{\theta} - \Theta_0)\}g, w_j),$$

$$(19)\qquad \frac{d}{dt}(\Theta^{(m)}(t), z_j) + ((v^{(m)}\cdot\nabla)\Theta^{(m)}, z_j)$$

$$= -\kappa(\nabla\Theta^{(m)},\nabla z_j) - ((v^{(m)}\cdot\nabla)\bar{\theta}, z_j) - (\bar{\theta}_t, z_j) - \kappa(\nabla\bar{\theta},\nabla z_j),$$

where $1 \leq j \leq m$. Moreover, for ${}^t(a, h-\bar{\theta}) \in H_\sigma(\Omega_n)\times L^2(\Omega_n)$

$$v^{(m)}(0) = v_{m0} = \sum_{j=1}^{m}(a, w_j)w_j, \tag{20}$$

$$\Theta^{(m)}(0) = \Theta_{m0} = \sum_{j=1}^{m}(h-\bar{\theta}(\cdot,0), z_j)z_j.$$

Multiplying (18) (resp. (19)) by $\alpha_{j,m}(t)$ (resp. $\beta_{j,m}(t)$), summing up with respect to j and noticing $((v^{(m)}\cdot\nabla)v^{(m)}, v^{(m)}) = 0$, $((v^{(m)}\cdot\nabla)\Theta^{(m)}, \Theta^{(m)}) = 0$, we have:

$$\frac{1}{2}\frac{d}{dt}\|v^{(m)}(t)\|^2 + \nu\|\nabla v^{(m)}(t)\|^2 = -(\alpha g\Theta^{(m)}, v^{(m)}) + ((1+\alpha\Theta_0)g, v^{(m)}) - (\alpha g\bar{\theta}, v^{(m)}), \tag{21}$$

$$\frac{1}{2}\frac{d}{dt}\|\Theta^{(m)}(t)\|^2 + \kappa\|\nabla\Theta^{(m)}(t)\|^2 = -((v^{(m)}\cdot\nabla)\bar{\theta}, \Theta^{(m)}) - (\bar{\theta}_t, \Theta^{(m)}) - \kappa(\nabla\bar{\theta}, \nabla\Theta^{(m)}). \tag{22}$$

Considering the assumption (A3) and Lemma 2.2, we have from (21)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v^{(m)}(t)\|^2 + \nu\|\nabla v^{(m)}(t)\|^2 &\leq 3\alpha\|g\|_{\frac{3}{2}}\cdot\|\Theta^{(m)}\|_6\cdot\|v^{(m)}\|_6 + (1+\alpha\Theta_0)\cdot \|g\|_{\frac{6}{5}}\cdot\|v^{(m)}\|_6 + 3\alpha\|g\|_{L^2(\Omega)}\cdot\|\bar{\theta}\|_3\cdot\|v^{(m)}\|_6 \\ &\leq \frac{3c^2\alpha\|g\|_{\frac{3}{2}}}{\sqrt{\kappa\nu}}\left(\frac{\kappa}{2}\|\nabla\Theta^{(m)}\|^2 + \frac{\nu}{2}\|\nabla v^{(m)}\|^2\right) + \frac{\nu}{4}\|\nabla v^{(m)}\|^2 + \frac{(1+\alpha\Theta_0)^2c^2}{\nu}\|g\|^2_{\frac{6}{5}} \\ &\quad + \frac{\nu}{4}\|\nabla v^{(m)}\|^2 + \frac{9c^2\alpha^2}{\nu}\|g\|_2^2\cdot\|\bar{\theta}\|_3^2,\end{aligned} \tag{23}$$

here $\|g\|_p = \|g\|_{L^p(\Omega)}$, $\|\bar{\theta}\|_p = \|\bar{\theta}\|_{L^p(\Omega)}$, $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega_n)}$, $c = (48)^{1/6}$. Then we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v^{(m)}(t)\|^2 + \frac{1}{2}\nu\|\nabla v^{(m)}(t)\|^2 &\leq \frac{3c^2\alpha\|g\|_{\frac{3}{2}}}{\sqrt{\kappa\nu}}\left(\frac{\kappa}{2}\|\nabla\Theta^{(m)}\|^2 + \frac{\nu}{2}\|\nabla v^{(m)}\|^2\right) \\ &\quad + \frac{(1+\alpha\Theta_0)^2c^2}{\nu}\|g\|^2_{\frac{6}{5}} + \frac{9\alpha^2c^2}{\nu}\|g\|_2^2\|\bar{\theta}\|_3^2.\end{aligned} \tag{24}$$

On the other hand, we have from (22)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta^{(m)}(t)\|^2 + \kappa\|\nabla\Theta^{(m)}\|^2 &\leq 3\|v^{(m)}\|_6\cdot\|\nabla\Theta^{(m)}\|\cdot\|\bar{\theta}\|_3 + 3\|\bar{\theta}_t\|_{\frac{6}{5}}\cdot \|\Theta^{(m)}\|_6 + \kappa\|\nabla\bar{\theta}\|\cdot\|\nabla\Theta^{(m)}\| \\ &\leq \frac{27c^2}{2\kappa}\|\bar{\theta}\|_3^2\cdot\|\nabla v^{(m)}\|^2 + \frac{27c^2}{2\kappa}\|\bar{\theta}_t\|^2_{\frac{6}{5}} + \frac{3}{2}\kappa\|\nabla\bar{\theta}\|^2 + 3\cdot\frac{\kappa}{6}\|\nabla\Theta^{(m)}\|^2,\end{aligned} \tag{25}$$

from which we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Theta^{(m)}(t)\|^2 + \frac{1}{2}\kappa\|\nabla\Theta^{(m)}\|^2 \leq \frac{27c^2}{2\kappa}\|\bar{\theta}\|_3^2\|\nabla v^{(m)}\|^2 + \frac{27c^2}{2\kappa}\|\bar{\theta}_t\|^2_{\frac{6}{5}} + \frac{3}{2}\kappa\|\nabla\bar{\theta}\|^2. \tag{26}$$

Adding (24) and (26), then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v^{(m)}(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\Theta^{(m)}(t)\|^2 + \frac{\nu}{2}\left(1 - \frac{3c^2\alpha\|g\|_{\frac{3}{2}}}{\sqrt{\kappa\nu}} - \frac{27c^2}{\kappa\nu}\|\bar{\theta}\|_3^2\right)\|\nabla v^{(m)}\|^2 \\ &+ \frac{\kappa}{2}\left(1 - \frac{3c^2\alpha\|g\|_{\frac{3}{2}}}{\sqrt{\kappa\nu}}\right)\|\nabla\Theta^{(m)}\|^2 \leq f(t),\end{aligned} \tag{27}$$

where $f(t) \equiv \frac{(1+\alpha\Theta_0)^2c^2}{\nu}\|g\|^2_{\frac{6}{5}} + \frac{9\alpha^2c^2}{\nu}\|g\|_2^2\cdot \|\bar{\theta}\|_3^2 + \frac{27c^2}{2\kappa}\|\bar{\theta}_t\|^2_{\frac{6}{5}} + \frac{3}{2}\kappa\|\nabla\bar{\theta}\|^2$.

Recalling the assumption of Theorem 3.2, we put $\gamma \equiv 1 - 3c^2\alpha\|g\|_{\frac{3}{2}}/\sqrt{\kappa\nu} > 0$. Furthermore thanks to (iv) of Lemma 2.1, we can take $\bar{\theta}$ such that $\sup_{0\leq t\leq T}\frac{27c^2}{\kappa\nu}\|\bar{\theta}(t)\|_3^2 \leq \frac{\gamma}{2}$. It is important for us that $\bar{\theta}$ can be taken in common not only in m but also for all $\Omega_n (n > 1)$. We put $\delta = \min\left\{\frac{\nu\gamma}{4}, \frac{\kappa\gamma}{2}\right\}$ (δ is independent of m and n). Then we have from (27)

$$\frac{d}{dt}(\|v^{(m)}(t)\|^2 + \|\Theta^{(m)}(t)\|^2) + 2\delta(\|\nabla v^{(m)}(t)\|^2 + \|\nabla\Theta^{(m)}(t)\|^2) \leq 2f(t). \tag{28}$$

Let d_n be a diameter of Ω_n. Owing to Poincaré's inequality, we find

$$\frac{d}{dt}(\|v^{(m)}(t)\|^2 + \|\Theta^{(m)}(t)\|^2) + \mu_n(\|v^{(m)}(t)\|^2 + \|\Theta^{(m)}(t)\|^2) \leq 2f(t), \tag{29}$$

where $\mu_n = (4\delta)/d_n^2$. Then we have from (29)

$$\begin{aligned}\|v^{(m)}(T)\|^2 + \|\Theta^{(m)}(T)\|^2 &\leq \exp(-\mu_nT)(\|v^{(m)}(0)\|^2 + \|\Theta^{(m)}(0)\|^2) \\ &\quad + 2\exp(-\mu_nT)\int_0^T \exp(\mu_nt)f(t)dt.\end{aligned} \tag{30}$$

Here we employ Brouwer's fixed point theorem. Indeed, in (17), we take initial values $\alpha_{j,m}(0)$, $\beta_{j,m}(0)$ $(j = 1,\cdots,m)$ as $(\alpha;\beta) = (\alpha_{1m},\cdots,\alpha_{mm},\beta_{1m},\cdots,\beta_{mm})$. Now we define a mapping $\boldsymbol{P}: \boldsymbol{R}^{2m}\to\boldsymbol{R}^{2m}$ as follows:

$$P((\alpha;\beta)) = (\alpha_{1,m}(T),\cdots,\alpha_{m,m}(T), \beta_{1,m}(T),\cdots,\beta_{m,m}(T)), \tag{31}$$

then it is easy to verify the mapping $\boldsymbol{P}$ is continuous. For $\lambda\in[0,1]$, we investigate possible solutions of the equation $(\alpha;\beta) = \lambda P((\alpha;\beta))$. In fact, we have by (30)

$$(32)\quad \begin{aligned} \|(\alpha;\beta)\|^2 &= \lambda^2 \|P((\alpha;\beta))\|^2 \\ &= \lambda^2 \|U^{(m)}(T)\|^2 \le \|U^{(m)}(T)\|^2 \\ &\le e^{-\mu_n T}\|U^{(m)}(0)\|^2 + 2e^{-\mu_n T}\int_0^T e^{\mu_n t} f(t)\,dt \\ &\le e^{-\mu_n T}\|(\alpha;\beta)\|^2 + \frac{2}{\mu_n}|||f|||(1-e^{-\mu_n T}), \end{aligned}$$

where $\|U^{(m)}(0)\|^2 = \|v^{(m)}(0)\|^2 + \|\Theta^{(m)}(0)\|^2$ and $|||f||| = \sup_{0\le t\le T} f(t)$. Since $\mu_n > 0$, we obtain $\|(\alpha;\beta)\|^2 \le \dfrac{2}{\mu_n}|||f|||$. Hence possible solutions $(\alpha;\beta)$ stay within a some definite ball. Therefore, thank to Brouwer's fixed point theorem, there is $(\alpha;\beta)$ satifying $(\alpha;\beta) = P((\alpha;\beta))$. This implies that there exists a periodic solution ${}^t(v^{(m)}, \Theta^{(m)})$ such that ${}^t(v^{(m)}(T), \Theta^{(m)}(T)) = {}^t(v^{(m)}(0), \Theta^{(m)}(0))$. We know by (32) the intial data which gives the periodic solution is in the ball $\left\{\|U^{(m)}(0)\|^2 \le \dfrac{2}{\mu_n}|||f|||\right\}$. On the other hand, from (28) we have

$$(33)\quad \begin{aligned} &\|v^{(m)}(t)\|^2 + \|\Theta^{(m)}(t)\|^2 + 2\delta\int_0^t (\|\nabla v^{(m)}(s)\|^2 + \|\nabla\Theta^{(m)}(s)\|^2)\,ds \\ &\le \|v^{(m)}(0)\|^2 + \|\Theta^{(m)}(0)\|^2 + 2\int_0^t f(s)\,ds \\ &\le \|v^{(m)}(0)\|^2 + \|\Theta^{(m)}(0)\|^2 + 2T|||f|||. \end{aligned}$$

Consequently, for m-dimensional periodic solutions ${}^t(v^{(m)}(t), \Theta^{(m)}(t))$, it holds that

$$(34)\quad \begin{aligned} &\|v^{(m)}(t)\|^2 + \|\Theta^{(m)}(t)\|^2 + 2\delta\int_0^t (\|\nabla v^{(m)}(s)\|^2 + \|\nabla\Theta^{(m)}(s)\|^2)\,ds \\ &\le 2\Big(\frac{1}{\mu_n} + T\Big)|||f||| \quad \text{for } m \ge 1. \end{aligned}$$

Therefore $\{v^{(m)}(t)\}_{m\ge1}$ (resp. $\{\Theta^{(m)}(t)\}_{m\ge1}$) is a bounded sequence in $L^2(0, T; H^1_\sigma(\Omega_n))$ (resp. $L^2(0, T; H^1_0(\Omega_n))$) and in $L^\infty_\pi(0, T; L^2(\Omega_n))$ (resp. $L^\infty_\pi(0, T; L^2(\Omega_n))$). Here a space $L^\infty_\pi(0, T; L^2(\Omega_n))$ means $\{u \in L^\infty(0, T; L^2(\Omega_n)); u(0) = u(T)\}$. Hence there exist subsequences $\{v^{(m)}\}$ and $\{\Theta^{(m)}\}$ (we used the same symbols) such that $v^{(m)} \to v$ (resp. $\Theta^{(m)} \to \Theta$) weakly in $L^2(0, T; H^1_\sigma(\Omega_n))$ (resp. $L^2(0, T; H^1_0(\Omega_n))$) and weakly* in $L^\infty_\pi(0, T; L^2(\Omega_n))$ (resp. $L^\infty_\pi(0, T; L^2(\Omega_n))$). Furthermore by using Lemma 2.3 (Friedrichs) and (34) we see that $v^{(m)} \to v$ and $\Theta^{(m)} \to \Theta$ strongly in $L^2(0, T; L^2(\Omega_n))$. Thanks to these facts, employing the usual argument of Galerkin's method, we can show that the limit function ${}^t(v, \Theta)$ is a periodic weak solution of (P_n) in Ω_n, and we skip it.

Moreover, we mention a lemma to prove Theorem 3.2.

Lemma 4.3. *Let ${}^t(v_n, \Theta_n)$ be a weak periodic solution for* (P_n) *obtained in Lemma* 4.2. *We put $u_n(x, t) = v_n(x, t)$ if $x \in \Omega_n$ and $u_n(x, t) = 0$ if $x \in \Omega\setminus\Omega_n$; $\theta_n(x, t) = \Theta_n(x, t)$ if $x \in \Omega_n$ and $\theta_n(x, t) = 0$ if $x \in \Omega\setminus\Omega_n$. Then $u_n \in L^2(0, T; V) \cap L^2_\pi(0, T; L^6(\Omega))$ and $\theta_n \in L^2(0, T; W) \cap L^2_\pi(0, T; L^6(\Omega))$. Moreover $\{u_n\}_{n\ge1}$* (resp. $\{\theta_n\}_{n\ge1}$) *is bounded in $L^2(0, T; V)$* (resp. $L^2(0, T; W)$) *and in $L^2_\pi(0, T; L^6(\Omega))$* (resp. $L^2_\pi(0, T; L^6(\Omega))$).

Proof of Lemma 4.3. We return to (28) and integrate it on $[0, T]$, then by virtue of the periodicity of $v^{(m)}(t)$ and $\Theta^{(m)}(t)$ with period T we get

$$(35)\quad \delta\int_0^T (\|\nabla v^{(m)}(t)\|^2 + \|\nabla\Theta^{(m)}(t)\|^2)\,dt \le \int_0^T f(t)\,dt \le T|||f|||,$$

where δ and $T|||f|||$ are independent of n and m. If we take $m \to \infty$ in (35), then we obtain by the lower semicontinuity of the norm with respect to the weak convergence

$$(36)\quad \delta\int_0^T (\|\nabla v_n(t)\|^2 + \|\nabla\Theta_n(t)\|^2)\,dt \le \int_0^T f(t)\,dt \le T|||f||| \quad (n \ge 1).$$

On the other hand, the equality $v_n(T) = v_n(0)$ in $L^2(\Omega_n)$ implies $v_n(T) = v_n(0)$ for a.e. $x \in \Omega_n$ and because of Lemma 2.2 we see $v_n(t) \in L^6(\Omega_n)$, therefore we find $v_n(T) = v_n(0)$ as elements of $L^6(\Omega_n)$. By this fact and (36) it holds that $v_n \in L^2_\pi(0, T; L^6(\Omega_n))$. Similarly we see $\Theta_n \in L^2_\pi(0, T; L^6(\Omega_n))$. Considering these results and using (36) again, it holds that for all $n \ge 1$, $u_n \in L^2(0, T; V) \cap L^2_\pi(0, T; L^6(\Omega))$, $\theta_n \in L^2(0, T; W) \cap L^2_\pi(0, T; L^6(\Omega))$ and (note $c = (48)^{1/6}$)

$$(37)\quad \frac{1}{c}\int_0^T (\|u_n(t)\|^2_{6(\Omega)} + \|\theta_n(t)\|^2_{L^6(\Omega)})\,dt \le \int_0^T (\|\nabla u_n(t)\|^2 + \|\nabla\theta_n(t)\|^2)\,dt \le \frac{T|||f|||}{\delta}.$$

Proof of Theorem 3.2. According to the uniform estimate (37), we can select subsequences $u_{n'}$, $\theta_{n'}$ and $u \in L^2(0, T; V) \cap L^2_\pi(0, T; L^6(\Omega))$, $\theta \in L^2(0, T; W) \cap L^2_\pi(0, T; L^6(\Omega))$ such that $u_{n'} \to u$ (resp. $\theta_{n'} \to \theta$) weakly in $L^2(0, T; V)$ (resp. $L^2(0, T; W)$) together with in $L^2_\pi(0, T;$

$L^6(\Omega))$ (resp. $L^2_\pi(0, T; L^6(\Omega)))$. Now we claim that there exist subsequences $u_{n'}$ and $\theta_{n'}$ such that for any bounded $\Omega' \subset \Omega$

(38) $u_{n'} \to u$ strongly in $L^2(0, T; L^2(\Omega'))$,

(39) $\theta_{n'} \to \theta$ strongly in $L^2(0, T; L^2(\Omega'))$.

We put $K_j = \bar{\Omega}_j$, then $\{K_j\}_{j=1}^\infty$ form a sequence of compact sets such that $K_1 \Subset K_2 \Subset \cdots \to \Omega (j \to \infty)$. Here, for each K_j we take $\alpha_j(x) \in C_0^\infty(\Omega)$ with the property $0 \leq \alpha_j \leq 1$, $\alpha_j|_{K_j} \equiv 1$, and $\operatorname{supp} \alpha_j \subset \Omega_{j+1}$. We note $K_j \subset \operatorname{supp} \alpha_j$. Here and after in this proof, $\|\cdot\|_{\Omega_j} = \|\cdot\|_{L^2(\Omega_j)}$, $d_j =$ the diameter of Ω_j. Then we construct a desired $\{u_{n'}\}$ as follows. First we make a sequence $\{\alpha_1(x) u_n(x)\}_{n=1}^\infty$, then this forms a uniformly bounded sequence of $L^2(0, T; W_0^{1,2}(\Omega_2))$. Indeed, noting $u_n(\Gamma) = 0$ and using Poincaré's inequality on Ω_2, then we see $\|\alpha_1 u_n\|_{\Omega_2} \leq \|u_n\|_{\Omega_2} \leq \frac{d_2}{\sqrt{2}} \|\nabla u_n\|_{\Omega_2}$. Hence we have by (37)

(40) $$\int_0^T \|\alpha_1 u_n\|^2_{\Omega_n} dt \leq \frac{d_2^2}{2} \int_0^T \|\nabla u_n\|^2 dt \leq \frac{d_2^2}{2} \frac{T|||f|||}{\delta}.$$

Moreover, $\|\nabla(\alpha_1 u_n)\|_{\Omega_2} \leq \|(\nabla\alpha_1) u_n\|_{\Omega_2} + \|\alpha_1(\nabla u_n)\|_{\Omega_2} \leq \left(\frac{d_2}{\sqrt{2}} \|\nabla \alpha_1\|_\infty + \|\alpha_1\|_\infty\right) \|\nabla u_n\|_{\Omega_2}$, where $\|w\|_\infty = \operatorname{ess.sup}_{x \in \Omega_2} |w(x)|$. Therefore we have

(41) $$\int_0^T \|\nabla(\alpha_1 u_n)\|^2_{\Omega_2} dt \leq \left(\frac{d_2}{\sqrt{2}} \|\nabla\alpha_1\|_\infty + \|\alpha_1\|_\infty\right)^2 \cdot \frac{T|||f|||}{\delta}.$$

By these estimates we find $\{\alpha_1 u_n\}_n$ is uniformly bounded in $L^2(0, T; W_0^{1,2}(\Omega_2))$. Consequently, there is a subsequence $\{\alpha_1 u_{1p}\}_{p=1}^\infty$ which converges weakly in $L^2(0, T; L_0^{1,2}(\Omega_2))$ and especially in $L^2(0, T; W^2(\Omega_2))$. Furthermore, according to Lemma 2.3, we get

(42) $$\int_0^T \|\alpha_1 u_{1p} - \alpha_1 u_{1q}\|^2_{\Omega_2} dt \leq \sum_{k=1}^{\ell_\varepsilon} \int_0^T (\alpha_1 u_{1p} - \alpha_1 u_{1q}, w_k)^2_{\Omega_2} dt + \varepsilon \int_0^T \|\alpha_1 u_{1p} - \alpha_1 u_{1q}\|^2_{W^{1,2}(\Omega_2)} dt$$
$$\leq \sum_{k=1}^{\ell_\varepsilon} \int_0^T (\alpha_1 u_{1p} - \alpha_1 u_{1q}, w_k)^2_{\Omega_2} dt + 4\varepsilon C_{\alpha_1} \frac{T|||f|||}{\delta} \to 4\varepsilon C_{\alpha_1} \frac{T|||f|||}{\delta},$$

as $p, q \to \infty$, where $C_{\alpha_1} = \frac{d_2^2}{2} + \left(\|\nabla\alpha_1\|_\infty \cdot \frac{d_2}{\sqrt{2}} + \|\alpha_1\|_\infty\right)^2$. As ε is arbitrary in (42), the sequence $\{\alpha_1 u_{1p}\}_{p=1}^\infty$ converges strongly in $L^2(0, T; L^2(\Omega_2))$. This implies that $\{u_{1p}\}_{p=1}^\infty$ converges strongly in $L^2(0, T; L^2(K_1))$. We repeat such an argument and we make $\{u_{jp}\}_{p=1}^\infty (j = 1, 2, \cdots)$. Choose diagonal components and denote them by $\{u_{n'}\}_{n'=1}^\infty$, then it converges on all K_j in $L^2(0, T; L^2(K_j))$ sense. As for $\{\theta_{n'}\}_{n'=1}^\infty$, we can show similarly.

Making use of (38) and (39), we can prove that ${}^t(u, \theta)$ is a periodic weak solution of (HCE). In fact, if we take an arbitrary test function ${}^t(\varphi, \psi)$, then we find a bounded domain Ω' and a number n_0 such that $\operatorname{supp} \varphi, \operatorname{supp} \psi \subset \Omega'$ and $\Omega' \subset \Omega_{n_0} \subset \Omega_n$ for all $n \geq n_0$. Then, with the aid of Lemma 2.2 and (37), we have

(43) $$\int_0^T |((u_{n'} \cdot \nabla)\varphi, u_{n'})_\Omega - ((u \cdot \nabla)\varphi, u)_\Omega| dt$$
$$\leq \int_0^T \{3\|u_{n'} - u\|_{L^2(\Omega')} \|u_{n'}\|_{L^6(\Omega)} \|\nabla\varphi\|_{L^3(\Omega')} + 3\|u\|_{L^6(\Omega)} \|u_{n'} - u\|_{L^2(\Omega')} \|\nabla\varphi\|_{L^3(\Omega')}\} dt$$
$$\leq 6c \cdot \left(\frac{T|||f|||}{\delta}\right)^{\frac{1}{2}} \|\nabla\varphi\|_{3,\infty} \cdot \left(\int_0^T \|u_{n'} - u\|^2_{L^2(\Omega')} dt\right)^{\frac{1}{2}} \to 0, \quad \text{as } n' \to \infty,$$

where $\|w\|_{3,\infty} = \sup_{0 \leq t \leq T} \|w(t)\|_{L^3(\Omega')}$. Similarly

(44) $$\int_0^T |((u_{n'} \cdot \nabla)\psi, \theta_{n'})_\Omega - ((u \cdot \nabla)\psi, \theta)_\Omega| dt$$
$$\leq \int_0^T \{3\|\theta_{n'} - \theta\|_{L^2(\Omega')} \|u_{n'}\|_{L^6(\Omega)} \|\nabla\psi\|_{L^3(\Omega')} + 3\|\theta\|_{L^6(\Omega)} \|u_{n'} - u\|_{L^2(\Omega')} \|\nabla\psi\|_{L^3(\Omega')}\} dt$$
$$\leq 3c \cdot \left(\frac{T|||f|||}{\delta}\right)^{\frac{1}{2}} \left\{\left(\int_0^T \|\theta_{n'} - \theta\|^2_{L^2(\Omega')} dt\right)^{\frac{1}{2}} + \left(\int_0^T \|u_{n'} - u\|^2_{L^2(\Omega')} dt\right)^{\frac{1}{2}}\right\} \|\nabla\psi\|_{3,\infty},$$

and the right hand side of (44) tends to 0 as $n' \to \infty$. We skip the remaining terms. Thus we have shown that ${}^t(u, \theta)$ is a periodic weak solution of (HCE).

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